Analytical theory of turbulent diffusion

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Recently Kraichnan (1959) has propounded a theory of homogeneous turbulence, based on a novel perturbation method, that leads to closed equations for the velocity covariance. In the present paper, this method is applied to the theory of turbulent diffusion and closed equations are derived for the probability distributions of the positions of marked fluid elements released in a turbulent flow.

Two topics are discussed in detail. The first is the probability distribution, at time t, of the displacement of an element from its initial position. In homogeneous flows, this distribution is found to resemble that for classical diffusion but with a variable coefficient of diffusion which is proportional to $v_0^2 t$ for $t \ll l/v_0$ and which approaches a constant value $\doteqdot lv_0$ for $t \gg l/v_0$ (l = macroscale, v_0 = r.m.s. turbulent velocity).

The second topic treated is the joint probability distribution of the displacements of two fluid elements. Particular attention is focused upon the probability distribution of relative displacement, i.e. Richardson's distance-neighbour function. This is found to be Gaussian for separations r which are large ($\gg l$). For smaller separations ($r \ll l$), its behaviour at high Reynolds numbers is found to be quite well expressed in terms of a variable diffusion coefficient $K(r,t)$, as suggested by Richardson (1926). For all but extremely short times, $K(r,t)$ is found to depend only on r and on the form of the inertial range spectrum $E(k)$. On assuming $E(k) \propto v_0^2 l(kl)^{-\frac{3}{2}}$ as results from Kraichnan's approximation (1959), one finds $K(r) \propto v_0 l(r/l)^{\frac{3}{2}}$. On the basis of similarity arguments of the Kolmogorov type, which give $E(k) \propto v_0^2 l(kl)^{-\frac{5}{3}}$, one finds $K(r) \propto v_0 l(r/l)^{\frac{4}{3}}$ as, in fact, Richardson originally proposed. The dispersion $\langle r^2 \rangle$ is proportional to $l^2(v_0 t/l)^4$ on Kraichnan's theory; while $\langle r^2 \rangle \propto l^2(v_0 t/l)^3$ on the similarity theory. This illustrates that the behaviour of $\langle r^2 \rangle$ is very sensitive to the spectrum.

1. Introduction

The aim of the theory of turbulent diffusion is to determine in a statistical sense the migration of marked particles as they are carried along with a turbulent flow. Like molecular motion in a dilute gas of discrete particles, turbulent diffusion is a linear process if the convected particles have no reaction on the flow; i.e. the probability distribution for the position of a marked particle in space obeys the superposition rule and changes in time according to a linear equation. Unlike classical molecular motion, the motion of neighbouring fluid elements in a

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continuum is correlated, although one expects that over distances large compared to the macroscale l of the turbulence this correlation is weak and that elements separated by such distances move almost independently. Furthermore, unlike classical molecular diffusion, turbulent diffusion is not a Markoff process. However, one expects that over times large compared to l/v_0, where v_0 is the root mean square fluid velocity, the fluid elements will suffer many essentially uncorrelated deflexions by the energy-containing eddies and that accordingly their displacement distribution will be almost Gaussian. Under such circumstances, one expects that the spread of marked particles carried by the fluid will indeed resemble classical diffusion, and that it will be possible to define a coefficient of eddy diffusivity. An analytical basis for these qualitative observations is given in § 2.

Particles which start out simultaneously at nearby points have closely similar histories in any one realization of the turbulent flow (and over times which are not too long). For such times, their relative motion is unaffected by eddies whose spacial scale is large compared to the initial separation; such eddies give nearly equal displacements to the two points. The change in the separation r is governed by the smaller eddies, particularly those whose length scale is of the same order as r. Thus, in a flow of the high Reynolds number, we expect that while the particles are separated by a distance appropriately small compared to l their relative diffusion will be governed by the inertial range spectrum of the turbulent flow, and will be unaffected by the structure of the energy-containing eddies. An analytical basis for these surmises is given in § 3, and a form is proposed for the variable diffusion coefficient $K(r)$ introduced by Richardson (1926). This form for $K(r)$ is very sensitive to the inertial range spectrum $E(k)$. On assuming that $E(k) \propto v_0^2 l(kl)^{-n}$, it is shown that $K(r) \propto v_0 l(r/l)^n$ and that, in consequence, $\langle r^2 \rangle \propto l^2(v_0 t/l)^{2/(2-n)}$. However, on the basic approximation from which these results are derived, the energy-containing eddies do play a part in the relative diffusion process. It is shown, however, that when modifications of the theory are made to exclude this effect, Kolmogoroff's spectrum $E(k) \propto v_0^2 l(kl)^{-\frac{5}{3}}$ implies $K(r) \propto v_0 l(r/l)^{\frac{4}{3}}$, as in fact Richardson (1926) originally proposed.

Turbulent diffusion is a simple example of turbulence dynamics and is therefore a suitable testing ground for examining the consequences of various approximations. Included in § 2.2 is a brief comparison between our basic analytical method and some alternative approximations.

2. Diffusion from a fixed source

2.1. *Methods of approach*

There are two main ways of attempting to give an analytical framework to the qualitative arguments of § 1. In the Lagrangian framework (as distinct from the Eulerian approach to be described presently), probability distributions are defined for the displacements, velocities, etc., of given marked particles, and the relationships between them are studied. For example, let $G(\mathbf{x}, t \mid \mathbf{x}_0, t_0)\, d\mathbf{x}$ be the probability that a fluid particle lying at the point $\mathbf{x}_0$ at time t_0 should, at the later time t, lie within a volume $d\mathbf{x}$ at the point $\mathbf{x}$. Let $V(\mathbf{x}, \mathbf{u}, t \mid \mathbf{x}_0, t_0)\, d\mathbf{x}\, d\mathbf{u}$ be the probability that this same particle should at that time lie within $d\mathbf{x}$ and have a velocity between $\mathbf{u}$ and $\mathbf{u}+d\mathbf{u}$. Then it is not difficult to show that

$$\frac{\partial}{\partial t}G(\mathbf{x},t\,|\,\mathbf{x}_0,t_0)+\frac{\partial}{\partial x_i}\int V(\mathbf{x},\mathbf{u},t\,|\,\mathbf{x}_0,t_0)\,u_i\,d\mathbf{u} = 0. \tag{2.1}$$

(In 2.1 and elsewhere, we use the summation convention.) Through the hydrodynamical equations, it is possible to derive a similar, though far more involved equation, relating V to another probability function, and this, in its turn, to yet another. It would be necessary to close this hierarchy of equations in some way in order to obtain from it an evaluation of $G(\mathbf{x},t\,|\,\mathbf{x}_0,t_0)$.

The second approach was first formulated by Batchelor (1952*b*); for a discussion of the general physical interpretation of the method the reader is referred to this paper, and for other applications to papers by Reid (1955) and Roberts (1957). The basic idea is one of reformulating the problem of finding Lagrangian probability functions as one of determining certain Eulerian moments. A passive scalar quantity $\psi(\mathbf{x},t)$ is introduced which satisfies the equation

$$\frac{\partial}{\partial t}\psi(\mathbf{x},t)+\frac{\partial}{\partial x_i}[\psi(\mathbf{x},t)\,u_i(\mathbf{x},t)] = 0, \tag{2.2}$$

where $u_i(\mathbf{x},t)$ is the velocity field. This quantity is therefore carried by the turbulent fluid, but does not affect its motion. It is clear that if we take

$$\psi(\mathbf{x},t_0) = \delta(\mathbf{x}-\mathbf{x}_0), \tag{2.3}$$

where $\delta(\mathbf{x})$ is the three-dimensional Dirac δ-function, then

$$\psi(\mathbf{x},t) = \delta(\mathbf{x}-\mathbf{x}_t), \tag{2.4}$$

where $\mathbf{x}_t$ is the position at time t (in this particular realization of $\mathbf{u}$) of the fluid element which was initially at $\mathbf{x}_0$. By averaging over all realizations, we see that the solution of

$$\frac{\partial}{\partial t}\langle\psi(\mathbf{x},t)\rangle+\frac{\partial}{\partial x_i}\langle\psi(\mathbf{x},t)\,u_i(\mathbf{x},t)\rangle = 0, \tag{2.5}$$

which satisfies the initial condition

$$\langle\psi(\mathbf{x},t_0)\rangle = \delta(\mathbf{x}-\mathbf{x}_0), \tag{2.6}$$

is

$$\langle\psi(\mathbf{x},t)\rangle = G(\mathbf{x},t\,|\,\mathbf{x}_0,t_0). \tag{2.7}$$

Through a similar though more complicated equation, $\langle\psi u_i\rangle$ is related to higher moments, and so on. Again, we have an hierarchy of equations which must be closed in some way in order to evaluate $G(\mathbf{x},t\,|\,\mathbf{x}_0,t_0)$.

In the present paper, we adopt the second of these two approaches and employ an approximation method for closing the equations devised by Kraichnan (1959). However, before presenting this approximate analysis, we shall derive some results which are asymptotically exact for small $t-t_0$.

2.2. *Exact results for short times*

For $t-t_0 \ll l/v_0$, the fluid particles are simply swept from their points of origin with whatever velocity the turbulent fluid happens to have at the moment of their release, i.e.

$$V(\mathbf{x},\mathbf{u},t\,|\,\mathbf{x}_0,t_0) = P[\mathbf{u}(\mathbf{x}_0,t_0)]\,\delta[(\mathbf{x}-\mathbf{x}_0)-\mathbf{u}(t-t_0)], \tag{2.8}$$

where $P[\mathbf{u}(\mathbf{x}_0, t_0)]$ is the probability density function (p.d.f.) of $\mathbf{u}$ at position $\mathbf{x}_0$ and time t_0. Thus, by (2.1), or by inspection,

$$G(\mathbf{x}, t \mid \mathbf{x}_0, t_0) = \frac{1}{(t-t_0)^3} P\left(\frac{\mathbf{x}-\mathbf{x}_0}{t-t_0}\right), \tag{2.9}$$

a result due to Batchelor (1952*a*).

This result can also be deduced from the formal solution

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x}, t_0) - \int_{t_0}^{t} dt' \frac{\partial}{\partial x_i} [u_i(\mathbf{x}, t')\, \psi(\mathbf{x}, t_0)] + \int_{t_0}^{t} dt' \frac{\partial}{\partial x_i} \left\{ u_i(\mathbf{x}, t') \int_{t_0}^{t'} dt'' \frac{\partial}{\partial x_j} [u_j(\mathbf{x}, t'')\, \psi(\mathbf{x}, t_0)] \right\} - \ldots, \tag{2.10}$$

which one obtains from (2.2) by integration and iteration. When $t - t_0 \ll l/v_0$, then $\mathbf{u}(\mathbf{x}, t) \doteqdot \mathbf{u}(\mathbf{x}_0, t_0)$, and it follows that

$$\psi(\mathbf{x}, t) \doteqdot \left[1 - (t-t_0)\, u_i(\mathbf{x}_0, t_0) \frac{\partial}{\partial x_i} + \frac{1}{2!} (t-t_0)^2\, u_i(\mathbf{x}_0, t_0)\, u_j(\mathbf{x}_0, t_0) \frac{\partial^2}{\partial x_i \partial x_j} - \ldots \right] \psi(\mathbf{x}, t_0). \tag{2.11}$$

Hence, by (2.6) and (2.7),

$$G(\mathbf{x}, t \mid \mathbf{x}_0, t_0) \doteqdot \left[1 - (t-t_0) \langle u_i(\mathbf{x}_0, t_0) \rangle \frac{\partial}{\partial x_i} + \frac{1}{2!} (t-t_0)^2 \langle u_i(\mathbf{x}_0, t_0)\, u_j(\mathbf{x}_0, t_0) \rangle \frac{\partial^2}{\partial x_i \partial x_j} - \ldots \right] \delta(\mathbf{x} - \mathbf{x}_0). \tag{2.12}$$

That this is equivalent to (2.9) can most easily be seen by expressing the result in wave-vector space, writing

$$G(\mathbf{x}, t \mid \mathbf{x}_0, t_0) = \int \tilde{G}(\mathbf{k}, t \mid \mathbf{x}_0, t_0)\, e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}_0)}\, d\mathbf{k}. \tag{2.13}$$

Then (2.12) is equivalent to

$$\tilde{G}(\mathbf{k}, t \mid \mathbf{x}_0, t_0) = \frac{1}{(2\pi)^3} \left[1 - ik_i(t-t_0) \langle u_i(\mathbf{x}_0, t_0) \rangle - \frac{1}{2!} k_i k_j (t-t_0)^2 \langle u_i(\mathbf{x}_0, t_0)\, u_j(\mathbf{x}_0, t_0) \rangle + \ldots \right]; \tag{2.14}$$

i.e. that is

$$\tilde{G}(\mathbf{k}, t \mid \mathbf{x}_0, t_0) = \frac{1}{(2\pi)^3} \tilde{P}[\mathbf{k}(t-t_0)], \tag{2.15}$$

where

$$\tilde{P}(\boldsymbol{\eta}) = \int P[\mathbf{u}(\mathbf{x}_0, t_0)]\, e^{-i\boldsymbol{\eta}.\mathbf{u}}\, d\mathbf{u} \tag{2.16}$$

is the characteristic function for the distribution of velocity at $\mathbf{x}_0$ and t_0. Equation (2.9) is simply the inverse of (2.15).

This second method of establishing the behaviour of G at small times brings out some noteworthy features. If (2.12) is cut off after any finite number of terms, it implies that $G(\mathbf{x}, t \mid \mathbf{x}_0, t_0)$ vanishes identically for non-zero $\mathbf{x} - \mathbf{x}_0$. On the other hand, if (2.14) is cut off after a finite number of terms, the resulting expression for $\tilde{G}$ diverges for large $\mathbf{k}$, or large $t - t_0$. *One concludes that any reasonable approximate solution for the full space-function $G(\mathbf{x}, t \mid \mathbf{x}_0, t_0)$ must include terms of all orders* *of the formal expansion* (2.10). Even for very short times the formal expansion is only useful because we happen to be able to sum it to all orders. However, if the moments

$$\langle \Delta x_i \rangle = \int \Delta x_i \, G(\mathbf{x}, t | \mathbf{x}_0, t_0) \, d\mathbf{x} = i(2\pi)^3 \left[\frac{\partial}{\partial k_i} \tilde{G}(\mathbf{k}, t \,|\, \mathbf{x}_0, t_0) \right]_{\mathbf{k}=0},$$

$$\langle \Delta x_i \, \Delta x_j \rangle = \int \Delta x_i \, \Delta x_j \, G(\mathbf{x}, t \,|\, \mathbf{x}_0, t_0) \, d\mathbf{x} = -(2\pi)^3 \left[\frac{\partial^2}{\partial k_i \, \partial k_j} \tilde{G}(\mathbf{k}, t \,|\, \mathbf{x}_0, t_0) \right]_{\mathbf{k}=0},$$

($\Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0$) are expanded by means of the formal solution, the resulting series appear to converge for all $t - t_0$, although the convergence is poor unless $t - t_0 \ll l/v_0$. Equation (2.14) shows that for small $t - t_0$,

$$\langle \Delta x_i \, \Delta x_j \rangle = U_{ij}(\mathbf{x}_0, t_0; \mathbf{x}_0, t_0) \, (t - t_0)^2, \tag{2.17}$$

where $$U_{ij}(\mathbf{x}, t; \mathbf{x}_0, t_0) = \langle u_i(\mathbf{x}, t) \, u_j(\mathbf{x}_0, t_0) \rangle$$

is the velocity covariance. In the isotropic case, therefore,

$$\langle \Delta x_i \, \Delta x_j \rangle = v_1^2 \, \delta_{ij} (t - t_0)^2, \tag{2.18}$$

where v_1^2 is the mean square of any component of velocity at position $\mathbf{x}_0$ and time t_0.

There is fairly strong experimental evidence (see, for example, Batchelor 1953, ch. 8) that P almost always is closely Gaussian. It follows, as Batchelor (1950) has pointed out, that $G(\mathbf{x}, t \,|\, \mathbf{x}_0, t_0)$ must be closely Gaussian for short times. Then by (2.14) and (2.9) we have

$$\tilde{G}(\mathbf{k}, t \,|\, \mathbf{x}_0, t_0) = \frac{1}{(2\pi)^3} \exp\left[-\tfrac{1}{2} U_{ij}(\mathbf{x}_0, t_0; \mathbf{x}_0, t_0) \, k_i \, k_j (t - t_0)^2 \right], \tag{2.19}$$

and $$G(\mathbf{x}, t \,|\, \mathbf{x}_0, t_0) = \frac{1}{(2\pi)^{\frac{3}{2}} (t - t_0)^3 \, (\det U_{ij})^{\frac{1}{2}}} \times \exp\left[-\tfrac{1}{2} u_{ij} \, \Delta x_i \, \Delta x_j (t - t_0)^{-2} \right], \tag{2.20}$$

where u_{ij} is the cofactor of $U_{ij}(\mathbf{x}_0, t_0; \mathbf{x}_0, t_0)$ and $\det U_{ij}$ denotes the determinant of these quantities. In the isotropic case,

$$\tilde{G}(\mathbf{k}, t \,|\, \mathbf{x}_0, t_0) = \frac{1}{(2\pi)^3} \exp\left[-\tfrac{1}{2} k^2 v_1^2 (t - t_0)^2 \right], \tag{2.21}$$

and $$G(\mathbf{x}, t \,|\, \mathbf{x}_0, t_0) = \frac{1}{[2\pi v_1^2 (t - t_0)^2]^{\frac{3}{2}}} \exp\left[-(\mathbf{x} - \mathbf{x}_0)^2 / 2 v_1^2 (t - t_0)^2 \right]. \tag{2.22}$$

2.3. *An integro-differential equation for* $G(\mathbf{x}, t | \mathbf{x}_0, t_0)$

When $t - t_0$ is not small compared to l/v_0, the approximation (2.11) to the formal solution (2.10) is invalid since it is no longer legitimate to ignore the space and time variation of $u_i(\mathbf{x}, t)$. It is nevertheless possible to effect a partial summation of (2.10) which includes terms from every order in the expansion, and is such that the resulting expression for $\tilde{G}$ converges for large $\mathbf{k}$. The integral equation for this approximate form of G can be derived in two different ways. The first make suse of Kraichnan's direct interaction approximation (Kraichnan 1959).

In the second method which has been discussed elsewhere (Roberts 1960), the same result is derived by discarding or retaining terms in the formal expansion according to a certain selection criterion. The terms retained are of all order. Each of these methods supposes that the velocity field is spatially homogeneous, but this restriction can be removed by the application of a more general method due to Kraichnan (1961). The final result, for the case where the mean field $\langle \mathbf{u}(\mathbf{x},t)\rangle$ vanishes, is the equation

$$\frac{\partial}{\partial t}G(\mathbf{x},t\mid \mathbf{x}_0,t_0) = \int_{t_0}^{t} dt' \int d\mathbf{x} U_{ij}(\mathbf{x},t;\,\mathbf{x}',t')\frac{\partial}{\partial x_i}G(\mathbf{x},t\mid \mathbf{x}',t')\frac{\partial}{\partial x'_j}G(\mathbf{x}',t'\mid \mathbf{x}_0,t_0). \tag{2.23}$$

Here, as later, the fluid velocity is supposed incompressible:

$$\partial u_i(\mathbf{x},t)/\partial x_i = 0. \tag{2.24}$$

(The compressible case can be treated by similar methods.) In the case of statistically homogeneous and stationary flows, we may write

$$\left.\begin{aligned} G(\mathbf{x},t\mid \mathbf{x}_0,t_0) &= G(\mathbf{x}-\mathbf{x}_0,t-t_0),\\ U_{ij}(\mathbf{x},t;\mathbf{x}_0,t_0) &= U_{ij}(\mathbf{x}-\mathbf{x}_0,t-t_0),\end{aligned}\right\} \tag{2.25}$$

and, upon a partial integration, (2.23) becomes

$$\frac{\partial}{\partial t}G(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j}\int_0^t dt' \int d\mathbf{x}' U_{ij}(\mathbf{x}',t')\,G(\mathbf{x}',t')\,G(\mathbf{x}-\mathbf{x}',t-t'). \tag{2.26}$$

This result—a consequence of Kraichnan's direct interaction approximation—is the central result of this section and much of this paper. To prove it, we first notice that, when the velocity field is spacially homogeneous, the problem of diffusion from a point source, although apparently possessing only radial symmetry even in the isotropic case, can always be rephrased as a homogeneous problem. For, since the equation (2.2) is linear, the response of the system to an initial disturbance

$$\psi(\mathbf{x},t_0) = e^{i\mathbf{k}.\mathbf{x}}, \tag{2.27}$$

is

$$\langle \psi(\mathbf{x},t)\rangle = \int d\mathbf{x}_0 G(\mathbf{x},t\mid \mathbf{x}_0,t_0)\,e^{i\mathbf{k}.\mathbf{x}_0}, \tag{2.28}$$

and, since for a homogeneous velocity field $G(\mathbf{x},t\mid \mathbf{x}_0,t_0)$ depends on $\mathbf{x}$ and $\mathbf{x}_0$ in the combination $\mathbf{x}-\mathbf{x}_0$ only, equation (2.28) can be rewritten

$$\langle \psi(\mathbf{x},t)\rangle = (2\pi)^3\,\tilde{G}(\mathbf{k},t\mid t_0)\,e^{i\mathbf{k}.\mathbf{x}}, \tag{2.29}$$

where

$$\tilde{G}(\mathbf{k},t\mid t_0) = \frac{1}{(2\pi)^3}\int G(\mathbf{x}-\mathbf{x}_0,t\mid t_0)\,e^{-i\mathbf{k}.(\mathbf{x}-\mathbf{x}_0)}\,d(\mathbf{x}-\mathbf{x}_0) \tag{2.30}$$

is the Fourier transform of the Green's function $G(\mathbf{x},t\mid \mathbf{x}_0,t_0)$. Equation (2.29) proves the average response matrix of the Fourier modes is diagonal (when the velocity field is homogeneous) and that $(2\pi)^3\tilde{G}(\mathbf{k},t\mid t_0)$ is the average response function for mode $\mathbf{k}$.

Having established this correspondence, we will now derive the approximate equation (2.26) for the response function by a method parallel to that employed by Kraichnan (1959)† for the velocity field response function. For simplicity, we will suppose henceforth that the velocity field is also statistically stationary. The modifications necessary if the field is not stationary are easily included.

It is convenient to introduce the artifice of cyclic boundary conditions over a large cube of side L in order to expand $\psi(\mathbf{x},t)$ and $\mathbf{u}(\mathbf{x},t)$ in Fourier sums rather than Fourier integrals:

$$\psi(\mathbf{x},t) = \sum_{\mathbf{k}} \tilde{\psi}(\mathbf{k},t)\, e^{i\mathbf{k}.\mathbf{x}} \quad [\tilde{\psi}(\mathbf{k}) = \tilde{\psi}^*(-\mathbf{k})], \tag{2.31}$$

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}} \tilde{\mathbf{u}}(\mathbf{k},t)\, e^{i\mathbf{k}.\mathbf{x}} \quad [\tilde{\mathbf{u}}(\mathbf{k}) = \tilde{\mathbf{u}}^*(-\mathbf{k})], \tag{2.32}$$

(cf. K., equation (2.1)). Equation (2.2) may be written

$$\frac{\partial \tilde{\psi}(\mathbf{k},t)}{\partial t} + ik_j \sum_{\mathbf{q}} \tilde{\mathbf{u}}_j(\mathbf{p},t)\, \tilde{\psi}(\mathbf{q},t) = 0, \tag{2.33}$$

where $\mathbf{p} = \mathbf{k}-\mathbf{q}$. The response function $\tilde{g}(\mathbf{k},t)$ for mode $\mathbf{k}$ is the solution of (2.33) under the initial conditions

$$\left.\begin{aligned} \tilde{\psi}(\mathbf{k},t) &\equiv \tilde{g}(\mathbf{k},t) = 1 \\ \tilde{\psi}(\mathbf{q},t) &\equiv \tilde{\psi}_{\mathbf{k}}(\mathbf{q},t) = 0 \quad (\mathbf{q} \neq \mathbf{k}) \end{aligned}\right\} t = 0, \tag{2.34}$$

(cf. § 2.1). By the equation of motion $\tilde{\psi}_{\mathbf{k}}(\mathbf{q},t)$ and the direct interaction approximation, we find (cf. K., equation (2.24)) that

$$\tilde{\psi}_{\mathbf{k}}(\mathbf{q},t) = -iq_l \int_0^t \tilde{u}_l(-\mathbf{p},t')\, \tilde{g}(\mathbf{k},t')\, \tilde{g}(\mathbf{q},t-t')\, dt'. \tag{2.35}$$

Thus by (2.33) we have

$$\frac{\partial \tilde{g}(\mathbf{k},t)}{\partial t} = -\sum_{\mathbf{q}} k_i k_j \int_0^t \tilde{u}_i(\mathbf{p},t)\, \tilde{u}_j(-\mathbf{p},t')\, \tilde{g}(\mathbf{q},t-t')\, \tilde{g}(\mathbf{k},t')\, dt', \tag{2.36}$$

and, on averaging, using the principle of weak statistical dependence (cf. K., § 2.2 and equation (2.25)), we find

$$\frac{\partial}{\partial t}\langle \tilde{g}(\mathbf{k},t)\rangle = -\sum_{\mathbf{q}} k_i k_j \int_0^t \langle \tilde{u}_i(\mathbf{p},t)\, \tilde{u}_j(-\mathbf{p},t')\rangle \langle \tilde{g}(\mathbf{q},t-t')\rangle \langle \tilde{g}(\mathbf{k},t')\rangle\, dt'. \tag{2.37}$$

Now let us take the limit. Make the transition $L \to \infty$. Let

$$\tilde{U}_{ij}(\mathbf{k},t-t') = \lim_{L\to\infty} \left(\frac{L}{2\pi}\right)^3 \langle \tilde{u}_i(\mathbf{k},t)\, \tilde{u}_j(-\mathbf{k},t')\rangle, \tag{2.38}$$

so that

$$U_{ij}(\mathbf{x}-\mathbf{x}',t-t') = \langle u_i(\mathbf{x},t)\, u_j(\mathbf{x}',t')\rangle = \int \tilde{U}_{ij}(\mathbf{k},t-t')\, e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}')}\, d\mathbf{k}, \tag{2.39}$$

and let

$$\tilde{G}(\mathbf{k},t) = \lim_{L\to\infty} \left(\frac{L}{2\pi}\right)^3 \langle \tilde{g}(\mathbf{k},t)\rangle \tag{2.40}$$

† This paper will be designated by 'K.' hereafter. Some differences in notation should be noted: In K., $g(\mathbf{k}, t)$ refers to the velocity field (impulse) response function and not to the response function for (2.33) below. Also, in K., g refers to an averaged response while, in this paper, it does not; the average being denoted by $\langle g\rangle$. Further, the notational distinction between a quantity and its Fourier transform is different from that adopted in this paper.

so that equations (2.30), (2.31) and (2.34) are consistent (cf. K., equations (3.2), (3.3)). Then

$$\frac{\partial \tilde{G}(\mathbf{k},t)}{\partial t} = -(2\pi)^3 k_i k_j \int_0^t dt' \int d\mathbf{q} \tilde{U}_{ij}(\mathbf{p},t') \tilde{G}(\mathbf{q},t') \tilde{G}(\mathbf{k},t-t'). \tag{2.41}$$

This result can be returned to physical space by writing it as

$$\frac{\partial \tilde{G}(\mathbf{k},t)}{\partial t} = -k_i k_j \int_0^t dt' \int d\mathbf{p} \int d\mathbf{q} \int d\mathbf{x}' \, \tilde{U}_{ij}(\mathbf{p},t') \tilde{G}(\mathbf{q},t') \tilde{G}(\mathbf{k},t-t') e^{i\mathbf{x}'.(\mathbf{p}+\mathbf{q}-\mathbf{k})}; \tag{2.42}$$

that is (cf. equations (2.30), (2.39))

$$\frac{\partial \tilde{G}(\mathbf{k},t)}{\partial t} = -k_i k_j \int_0^t dt' \int d\mathbf{x}' U_{ij}(\mathbf{x}',t') G(\mathbf{x}',t') \tilde{G}(\mathbf{k},t-t') e^{-i\mathbf{k}.\mathbf{x}'}. \tag{2.43}$$

On using equation (2.30) again, we recover equation (2.26).

We will now investigate some elementary consequences of equation (2.26). On multiplying each side by $x_i x_j$ and integrating over all $\mathbf{x}$ and the right-hand side by parts, we find

$$\frac{\partial}{\partial t}\langle x_i x_j \rangle = \int x_i x_j \frac{\partial}{\partial t} G(\mathbf{x},t)\, d\mathbf{x} = 2\int_0^t dt' \int d\mathbf{x}' U_{ij}(\mathbf{x}',t') G(\mathbf{x}',t');$$

that is

$$\langle x_i x_j \rangle = 2t\kappa_{ij}(t), \tag{2.44}$$

where $\kappa_{ij}(t)$ is defined by

$$\kappa_{ij}(t) = \frac{1}{t}\int_0^t dt'(t-t') \int d\mathbf{x}' U_{ij}(\mathbf{x}',t') G(\mathbf{x}',t'). \tag{2.45}$$

For short times ($t \ll l/v_0$), (2.44) and (2.45) agree with (2.17) if we assume $G(\mathbf{x}',t')$ is negligible unless $|\mathbf{x}'| \ll l$. Then,

$$\kappa_{ij} \doteqdot \tfrac{1}{2} t U_{ij}(0,0) \quad (t \to 0). \tag{2.46}$$

For large times ($t \gg l/v_0$),

$$\kappa_{ij} \doteqdot \int_0^\infty dt' \int d\mathbf{x}' U_{ij}(\mathbf{x}',t') G(\mathbf{x}',t') = \frac{1}{(2\pi)^3}\int_0^\infty dt' \int d\mathbf{k} \tilde{U}_{ij}(\mathbf{k},t') \tilde{G}(-\mathbf{k},t')$$
$$= \kappa_{ij}^\infty, \quad \text{say.} \tag{2.47}$$

Thus, for large times, $\kappa_{ij}(t)$ can be recognized as an effective 'eddy diffusivity'. Let us assume that the diffusion for time $\gtrsim l/v_0$ is dominated by the energy-containing eddies. Then it is reasonable to suppose that the integrals (2.47) should depend only on the parameters l and v_0, whence, by dimensional reasoning, we must have

$$\kappa_{ij}^\infty = \text{constant of order } l v_0.$$

In the isotropic case, we have (setting $x = |\mathbf{x}|$)

$$\kappa_{ij} = \kappa \delta_{ij}, \quad \kappa = \frac{4\pi}{3t}\int_0^t dt'(t-t') \int_0^\infty dx' x'^2 U_{ii}(x',t') G(x',t'), \tag{2.48}$$

and

$$\kappa_{ij}^\infty = \kappa^\infty \delta_{ij}, \quad \kappa^\infty = \frac{4\pi}{3}\int_0^\infty dt' \int_0^\infty dx' x'^2 U_{ii}(x',t') G(x',t'). \tag{2.49}$$

Equations (2.44) and (2.45) and their generalizations for flow which are not steady or homogeneous, imply that the Lagrangian correlation function $\langle u_i(t)\, u_j(t_0)\rangle_L = U_{Lij}(t-t_0)$ (say) is

$$U_{Lij}(t-t_0) = \int d\mathbf{x} U_{ij}(\mathbf{x},t;\mathbf{x}_0,t_0)\, G(\mathbf{x},t\,|\,\mathbf{x}_0,t_0). \qquad (2.50)$$

This should be compared to the exact result

$$U_{Lij}(t-t_0) = \int d\mathbf{x} \langle u_i(\mathbf{x},t)\, u_j(\mathbf{x}_0,t_0)\, \psi(\mathbf{x},t\,|\,\mathbf{x}_0,t_0)\rangle, \qquad (2.51)$$

where $\psi(\mathbf{x},t\,|\,\mathbf{x}_0,t_0)$ denotes the unaveraged Green's function, i.e. the solution (2.4) of (2.2) satisfying (2.3). Equation (2.50) clearly may be obtained from (2.51) on the assumption that the ψ and $\mathbf{u}$ fields are statistically independent (see also Corrsin 1960).† However, if one decides to use an assumption of this kind, the results depend very much on what stage of the analysis is chosen for its application. For example, had we assumed that ψ and $\mathbf{u}$ are uncorrelated in (2.2), we would have obtained the absurd result $\partial\langle\psi(\mathbf{x},t)\rangle/\partial t = 0$. Moreover, if the approximation

$$\langle u_i(\mathbf{x},t)\, u_j(\mathbf{x}_0,t_0)\, \psi(\mathbf{x},t\,|\,\mathbf{x}_0,t_0)\rangle = U_{ij}(\mathbf{x},t;\mathbf{x}_0,t_0)\, G(\mathbf{x},t\,|\,\mathbf{x}_0,t_0) \qquad (2.52)$$

is used to close the hierarchy of equations which arise from (2.2) and (2.3), the result is *not* consistent with (2.50) (cf. Roberts 1957). Thus, the direct-interaction approximation is consistent with (2.52) only if (2.52) is used in a particular way. It is not clear why this should be so. However, Bourret (1960), in studying a model of turbulent diffusion due to Taylor (1922) (and which is described briefly below), has commented that, although the particle displacement cannot be Markovian, it may not be unreasonable to suppose that the particle velocity is Markovian. If this is the case, it is perhaps not altogether surprising that, when the quasi-normality approximation is applied directly to the equations for $G(\mathbf{x},t)$, it should give worse results than when applied† to the Lagrangian velocity correlation.

Equation (2.26) is non-local in space and time, in contrast to an ordinary diffusion equation:

$$\frac{\partial G(\mathbf{x},t)}{\partial t} = \kappa_{ij}^{\infty}\frac{\partial^2 G(\mathbf{x},t)}{\partial x_i \partial x_j}. \qquad (2.53)$$

Equation (2.53) describes molecular diffusion on time and distance scales large compared to the collision time and the mean free path, respectively. Over these large times and distances, molecular diffusion may be thought of as a random walk process for molecules. In like manner, turbulent convection may be thought of as a random walk process for fluid elements in which the effective step length is $\sim l$, and the effective velocity is $\sim v_0$. In fact, we shall see in § 2.5 that, on length and time scales large compared to l and l/v_0 respectively, (2.26) does reduce to the form (2.53). On smaller scales, however, the equation governing $G(\mathbf{x},t)$ must reflect the persistence of correlations over finite space and time intervals.

† Saffman (1959) has used the approximate result (2.50) to derive an estimate for κ (see § 2.5 below).

Batchelor & Townsend (1956, §2, p. 360) first suggested that this non-localness may be best expressed by an integro-differential equation. Another such equation has been derived by Bourret (1960) by generalizing a property of a simple model of turbulent diffusion due to Taylor (1922) which incorporates a persistence of velocity correlation along the path of the diffusing element, i.e. a finite Lagrangian correlation time. Taylor supposed that the fluid elements moved with velocity $\pm v$ between the equally spaced points of a lattice on the x-axis, and that the motion of an element could only be reversed at a lattice point, the probability of such an event being q. Following Goldstein (1951), who made a detailed study of Taylor's model, Bourret considered the limiting case $q \to 0$, $d \to 0$, $d/q \to l$, where d is the lattice separation. He proved that, for Taylor's model, $G(\mathbf{x}, t)$ satisfied an integro-differential equation whose generalization to three dimensions is

$$\frac{\partial G(\mathbf{x},t)}{\partial t} = \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t U_{Lij}(t-t')\, G(\mathbf{x},t')\, dt'. \tag{2.54}$$

Bourret made the hypothesis that (2.54) is not a property of Taylor's model alone, but is more generally valid.

We may write (2.26) in the form

$$\frac{\partial G(\mathbf{x},t)}{\partial t} = \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t dt' \int d\mathbf{x}' Q_{ij}(\mathbf{x}-\mathbf{x}', t-t')\, G(\mathbf{x}',t'), \tag{2.55}$$

where, according to our application of Kraichnan's approximation,

$$Q_{ij}(\mathbf{x},t) \doteqdot G(\mathbf{x},t)\, U_{ij}(\mathbf{x},t).$$

Equation (2.55) bears a strong formal resemblance to Bourett's result. Like (2.55), equation (2.54) is non-local in time but, unlike (2.55), it is local in space. This seems to imply that, although (2.54) takes into account the persistence of velocity correlation along the path of a fluid element, it does not take into account the fact that a cloud of marked particles set down at random in a turbulent flow must also display velocity correlation in space at any instant of time. The fact that it does not do so is not surprising since the model from which (2.54) was derived also does not incorporate this spatial velocity correlation. It would appear that such a correlation is contained in (2.55) through the spacial non-localness of that equation. This conclusion is supported by a study of a generalization of Taylor's model (Roberts 1961); we consider two fluid elements in motion along the lattice defined in Taylor's model, and suppose that the probability of the reversal of the velocity of either element on encountering a lattice point depends on whether the two elements were moving in the same or in opposite directions immediately before the encounter. It is found that (2.55) is obeyed and that $Q_{ij}(\mathbf{x}, t)$ is the Green function for the equation governing the diffusive flux. It is also found that (2.54) is obeyed if, and only if, the motion of the two elements is uncorrelated.

2.4. *Solution of the equation for short times*

For $t \ll l/v_0$, we may assume that the G factors in the integrand of (2.23) are negligible unless $\mathbf{x}-\mathbf{x}_0$ and $\mathbf{x}'-\mathbf{x}_0$ are both small compared with l. Thus,

$U_{ij}(\mathbf{x}', t', \mathbf{x}_0, t_0)$ in the integrand may be replaced by $U_{ij}(\mathbf{x}_0, t_0, \mathbf{x}_0, t_0)$ or, in the isotropic case, by $v_1^2 \delta_{ij}$. It follows that

$$\frac{\partial}{\partial t} G(\mathbf{x}, t) = v_1^2 \nabla^2 \int_0^t dt' \int d\mathbf{x}' G(\mathbf{x}', t') G(\mathbf{x}-\mathbf{x}', t-t'). \tag{2.56}$$

Taking the Fourier transform

$$G(\mathbf{x}, t) = \int \tilde{G}(\mathbf{k}, t') e^{i\mathbf{k}.\mathbf{x}} d\mathbf{k},$$

we find

$$\frac{\partial}{\partial t} \tilde{G}(\mathbf{k}, t) = -(2\pi)^3 v_1^2 k^2 \int_0^t dt' \tilde{G}(\mathbf{k}, t') \tilde{G}(\mathbf{k}, t-t'). \tag{2.57}$$

By taking the Laplace transform of (2.57), or by using a result of Watson (1944, §12.2, equation (5)), we find

$$\tilde{G}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \frac{J_1(2kv_1 t)}{kv_1 t} \quad (k = |\mathbf{k}|). \tag{2.58}$$

Inverting by using a further result of Watson (1944, §13.42, equation (4)), we find

$$G(\mathbf{x}, t) = \begin{cases} \dfrac{1}{(2\pi v_1 t)^2} [4v_1^2 t^2 - x^2]^{-\frac{1}{2}} & \text{if} \quad x < 2v_1 t, \\ 0 & \text{if} \quad x > 2v_1 t. \end{cases} \tag{2.59}$$

The corresponding probability distribution $D(x_1, t)$ for displacement along the x_1-axis (whose direction may be chosen arbitrarily) is related to $G(x, t)$ by

$$\partial D(x, t)/\partial x = -2\pi x G(x, t), \tag{2.60}$$

and so, in the present case,

$$D(x_1, t) = \begin{cases} \dfrac{1}{2\pi (v_1 t)^2} [4v_1^2 t^2 - x_1^2]^{\frac{1}{2}} & \text{if} \quad x_1 < 2v_1 t, \\ 0 & \text{if} \quad x_1 > 2v_1 t. \end{cases} \tag{2.61}$$

Both $G(x, t)$ and $D(x_1, t)$ are everywhere non-negative, but they are zero beyond a distance of $2v_1 t$ from the source. That this behaviour is not restricted to small times can be proved directly from equation (2.26) by induction. The finite maximum 'propagation speed' exhibited by equation (2.59) therefore persists for all times. This is also consistent with equation (2.9) which implies, in the present case, that the p.d.f. of velocity has a sharp cut-off at $u = 2v_1$. In fact, this p.d.f. predicted by the direct interaction approximation agrees with the actual p.d.f. of velocity only as far as the second moments.

A related unrealistic feature of (2.59) is that $G(x, t) \to \infty$ as $x \to 2v_1 t - 0$. However, the singularity is weak, since

$$\int_x^{2v_1 t + 0} 4\pi x^2 G(x, t)\, dx \doteqdot \frac{4}{\pi} \left(2 - \frac{x}{v_1 t}\right)^{\frac{1}{2}} \quad (x \to 2v_1 t - 0);$$

that is, the singular outgoing wave-front does not even carry a finite integrated probability. This singular type of behaviour may be attributed to the combined effects of persistence of velocity and finite maximum propagations peed. Another example is provided by Taylor's model (Taylor 1922). In this case the singular outgoing wave-front even carries a finite integrated probability (cf. Goldstein 1951, § 8).

For a normally distributed velocity field, a comparison of equation (2.58) in the form

$$\tilde{G}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\sum_0^\infty \frac{(-1)^n (kv_1 t)^{2n}}{n!(n+1)!}, \tag{2.62}$$

with the exact solution (2.21) in the form

$$\tilde{G}(\mathbf{k},t) = \frac{1}{(2\pi)^3}\sum_0^\infty \frac{(-1)^n (kv_1 t)^{2n}}{2^n n!}, \tag{2.63}$$

shows that, for short times, the direct interaction approximation gives the second moments of $G(\mathbf{x},t)$ correctly. The fractional errors in the fourth, sixth, and $2n$th moments are, respectively, $\frac{1}{3}$, $\frac{2}{3}$, and $[1-2^n/(n+1)!]$. This exhibits in another way the consequence of the effective cut-off in 'propagation speed'.

2.5. *Solution for large times*

For $t \gg l/v_0$, the diffusing particles will have suffered many displacements (statistically almost unrelated) from the energy-containing eddies, and we may expect $G(\mathbf{x},t)$ to become close to a Gaussian distribution. Batchelor & Townsend (1956, p. 358) have shown that this can be established, granted the truth of an as yet unproved extension of the central limit theorem. Alternatively, we may proceed directly by expressing the moments of the distribution of $\mathbf{x}$ as integrals over the Lagrangian velocity correlations. As Taylor (1922) has shown

$$\langle x_i x_j\rangle \to 2\kappa^*_{ij} t \quad (t\to\infty), \tag{2.64}$$

where

$$\kappa^*_{ij} = \int_0^\infty dt' U_{Lij}(t'). \tag{2.65}$$

Let us apply the argument used by Taylor to derive this result to the evaluation of a fourth moment such as $\langle x_i x_j x_k x_l\rangle$. We find

$$\langle x_i x_j x_k x_l\rangle = \int_0^t dt_1 \int_0^t dt_2 \int_0^t dt_3 \int_0^t dt_4 \langle u_i(t_1)\, u_j(t_2)\, u_k(t_3)\, u_l(t_4)\rangle_L. \tag{2.66}$$

Let τ be the Lagrangian velocity correlation time (assumed finite). It is clear that the integrand of (2.66) is negligibly small *unless*

either $\quad |t_1-t_2| < \tau$ *and* $|t_3-t_4| < \tau$,

or $\quad |t_1-t_3| < \tau$ *and* $|t_4-t_2| < \tau$,

or $\quad |t_1-t_4| < \tau$ *and* $|t_2-t_3| < \tau$,

or $\quad |t_\alpha-t_\beta| < \tau$, *all* $\alpha,\beta = 1$–4.

For $t \gg \tau$, the region defined by the last of these inequalities makes a contribution to $\langle x_i x_j x_k x_l\rangle$ of the order of

$$\int_0^t dt_1 \int_{-\tau}^{\tau} ds_1 \int_{-\tau}^{\tau} ds_2 \int_{-\tau}^{\tau} ds_3 \langle u_i(t_1)\, u_j(t_1+s_1)\, u_k(t_1+s_2)\, u_l(t_1+s_3)\rangle_L.$$

The integral over s_1, s_2, s_3 is bounded and independent of t_1. Thus the contribution made by this region to $\langle x_i x_j x_k x_l\rangle$ is of order t, $t\to\infty$. However, the ranges of

t_1 to t_4 for which the first of the inequalities *alone* is satisfied, makes a contribution to $\langle x_i x_j x_k x_l \rangle$ which is of order t^2, $t \to \infty$. It is

$$\left[\int_0^t dt_1 \int_0^t dt_2 \langle u_i(t_1)\, u_j(t_2)\rangle\right]\left[\int_0^t dt_3 \int_0^t dt_4 \langle u_k(t_3)\, u_l(t_4)\rangle\right],$$

which is equal to $\langle x_i x_j\rangle \langle x_k x_l\rangle$. The regions defined by the other inequalities make similar contributions. It follows that, for $t \gg \tau$,

$$\langle x_i x_j x_k x_l\rangle = \langle x_i x_j\rangle\langle x_k x_l\rangle + \langle x_i x_k\rangle\langle x_l x_j\rangle + \langle x_i x_l\rangle\langle x_j x_k\rangle,$$

that is, the fourth cumulants are negligible if $t \gg \tau$. Similar arguments hold for moments of all orders, showing that $G(\mathbf{x}, t)$ approaches normality as $t \to \infty$.

By a proof which is an exact parallel to the above, we can show that, for $t \gg l/v_0$, the distribution satisfying (2.26) approaches normality and that the corresponding eddy diffusivity is

$$\kappa_{ij}^{\infty} = \int_0^{\infty} dt' \int d\mathbf{x}' G(\mathbf{x}', t')\, U_{ij}(\mathbf{x}', t'). \tag{2.67}$$

Consider, for example, the fourth moment $\langle x_i x_j x_k x_l\rangle$. It is easy to show from (2.26) that

$$\langle x_i x_j x_k x_l\rangle = (x_i x_j\rangle\langle x_k x_l\rangle + \langle x_i x_k\rangle\langle x_j x_l\rangle + \langle x_i x_l\rangle\langle x_j x_k\rangle$$
$$+2\left[\int_0^t dt'(t-t')\int d\mathbf{x}' x_i' x_j' U_{kl}(\mathbf{x}', t')\, G(\mathbf{x}', t') + 5\,\text{similar terms}\right]. \tag{2.68}$$

Assuming that expressions such as

$$\frac{1}{t}\int_0^t dt'(t-t')\int d\mathbf{x}' x_i' x_j' U_{kl}(\mathbf{x}', t')\, G(\mathbf{x}', t')$$

converge as $t \to \infty$, we see that for $t \gg l/v_0$ the first three terms on the right-hand side of (2.68) are of order $(lv_0 t)^2$ while the remainder are of order $l^3 v_0 t$ and therefore are comparatively negligible. Thus the fourth-order cumulants can be neglected if $t \gg l/v_0$. Similar arguments hold in all orders and show that G approaches normality. This result may also be established by the following alternative method.

If $t \gg l/v_0$, the mean distance $\sqrt{\langle x^2\rangle}$ the particles will have travelled from their source and will be large compared to l so that the length and time scales of $G(\mathbf{x}, t)$ will be large compared to l and l/v_0, respectively. Under these circumstances, the only regions of integration in (2.26) for which the integrand is appreciable are those for which $G(\mathbf{x}-\mathbf{x}', t-t')$ is approximately equal to $G(\mathbf{x}, t)$. Thus, with κ_{ij}^{∞} given by (2.67), we have

$$\frac{\partial G(\mathbf{x}, t)}{\partial t} = \kappa_{ij}^{\infty}\frac{\partial^2 G(\mathbf{x}, t)}{\partial x_i \partial x_j}, \tag{2.69}$$

since for such large times it is immaterial whether the upper limit of integration in (2.26) is t or ∞. In the isotropic case (2.69) becomes

$$\frac{\partial G(\mathbf{x}, t)}{\partial t} = \kappa^{\infty}\nabla^2 G(\mathbf{x}, t), \tag{2.70}$$

where κ^{∞} is given by (2.49).

We may regard (2.69) as the first term in a series of approximations based on expanding the term $G(\mathbf{x}-\mathbf{x}', t-t')$ in the integrand of (2.26) in a Taylor series about the point $(\mathbf{x}, t)$, and we may apply an *a posteriori* check on the reasonableness of the approximation by verifying that, on substituting in the second term of this series the value of $G(\mathbf{x}, t)$ derived from (2.69), this second term is small compared to either side of (2.69). We will refrain from giving the analysis which is straightforward. It confirms that, if the relevant integrals converge, the necessary conditions for the validity of (2.69) are

$$t \gg l/v_0, \quad x \ll v_0 t. \tag{2.71}$$

The second condition arises from the finiteness of the maximum propagation velocity which, as we have already seen, gives an artificial cut-off at the distance $2v_1 t$. We may expect that for the actual case in which this cut-off is not present the second of the conditions (2.71) would be unnecessary and that the distribution of particles would be Gaussian at all distances. In any event, when the first condition is satisfied, it is clear that the fraction of particles affected by the second condition is negligibly small.

It seems extremely likely, from the exact short-time result of § 2.2 and our present results for large times, that $G(\mathbf{x}, t)$ is nearly Gaussian for all times, and that the variably diffusion coefficients defined in §2.3 (cf. equations (2.44), (2.45) and (2.48)) will give useful estimates of the variance of the distribution at all times. On assuming a Gaussian form for $G(\mathbf{x}, t)$, we obtain from (2.45) an integral equation for $\kappa_{ij}(t)$. The approximation of (2.51) by (2.50) has been proposed independently by P. G. Saffman (unpublished) in the homogeneous case. By assuming a Gaussian form for $G(\mathbf{x}, t)$, and the isotropic form

$$\breve{U}_{ij}(\mathbf{k}, t) = \left[\delta_{ij} - \frac{k_i k_j}{k^2}\right] \frac{v_1^2 k^2}{\pi^{\frac{3}{2}} k_0^5} \exp\left\{-\left[\frac{k^2}{k_0^2} + \tfrac{1}{2} v_1^2 k^2 t^2\right]\right\}, \tag{2.72}$$

he has obtained from the integral equation for κ_{ij} the estimate

$$\kappa^{\infty} \sim 0{\cdot}7 v_1/k_0, \tag{2.73}$$

where $\pi^{\frac{1}{2}} k_0^{-1}$ is the longitudinal integral scale (cf. Batchelor 1953, p. 47).

3. Relative diffusion

3.1. *Formulation of problem: exact results for short times*

In this section, we study the correlation between the motion of two marked particles which are initially separated by a distance small compared to l. The choice of method is essentially that of § 2.1 and again we will adopt a formulation in terms of Eulerian moments. We introduce the passive scalar field $\psi_1(\mathbf{x}, t)$ for the first particle and $\psi_2(\mathbf{y}, s)$ for the second particle, and we require that both fields satisfy (2.2). For $\psi_1(\mathbf{x}, t)$, we take as initial condition

$$\psi_1(\mathbf{x}, t_0) = \delta(\mathbf{x}-\mathbf{x}_0), \tag{3.1}$$

and, for $\psi_2(\mathbf{x}, s)$, we take

$$\psi_2(\mathbf{y}, s_0) = \delta(\mathbf{y}-\mathbf{y}_0). \tag{3.2}$$

Let us define

$$B(\mathbf{x}, t; \mathbf{y}, s \mid x_0, t_0; \mathbf{y}_0, s_0) = \langle \psi_1(\mathbf{x}, t)\, \psi_2(\mathbf{y}, s)\rangle. \tag{3.3}$$

This is the joint p.d.f. of particle displacements $\mathbf{x}$ and $\mathbf{y}$ at times t and s from positions $\mathbf{x}_0$ and $\mathbf{y}_0$ at initial instants t_0 and s_0, respectively. We shall term this 'the two particle Green's function'. Clearly, since the two particles have identical properties,

$$B(\mathbf{x},t;\mathbf{y},s|\mathbf{x}_0,t_0;\mathbf{y}_0,s_0) = B(\mathbf{y},s;\mathbf{x},t|\,\mathbf{y}_0,s;\mathbf{x}_0,t_0).$$

In homogeneous steady turbulence, it depends only on difference times and difference co-ordinates. We shall then write it as

$$B(\mathbf{x}-\mathbf{x}_0,t-t_0;\mathbf{y}-\mathbf{y}_0,s-s_0\,|\,\mathbf{r}_0,\tau_0) = B(\mathbf{y}-\mathbf{y}_0,s-s_0;\mathbf{x}-\mathbf{x}_0,t-t_0\,|\,-\mathbf{r}_0,-\tau_0),$$

where $\mathbf{r}_0 = \mathbf{y}_0-\mathbf{x}_0$ and $\tau_0 = s_0-t_0$.

It is evident that if one integrates (3.3) over all $\mathbf{x}$ or all $\mathbf{y}$ the one-point Green's function of § 2 is recovered. Of more interest is the integral

$$R(\mathbf{r},t,s\,|\,\mathbf{x}_0,\mathbf{r}_0,t_0,s_0) = \int d\mathbf{x}B(\mathbf{x},t;\mathbf{x}+\mathbf{r},s\,|\,\mathbf{x}_0,t_0;\mathbf{x}_0+\mathbf{r}_0,s_0). \tag{3.4}$$

$R(\mathbf{r},t,t\,|\,\mathbf{x}_0,\mathbf{r}_0,t_0,t_0)$ is Richardson's 'distance neighbour function' (Richardson 1926). It denotes the p.d.f. at time t of separation $\mathbf{r}$ for a pair of particles which at time t_0 were situated at $\mathbf{x}_0$ and $\mathbf{x}_0+\mathbf{r}_0$. For the homogeneous and stationary flows with which we will be primarily concerned, R depends only on $\mathbf{r}_0,\tau_0,\mathbf{r}-\mathbf{r}_0,t-t_0$, and $s-s_0$ and will be written†

$$R(\mathbf{r}-\mathbf{r}_0,t-t_0,s-s_0\,|\,\mathbf{r}_0,\tau_0) = \int d\mathbf{x}B(\mathbf{x}-\mathbf{x}_0,t-t_0;(\mathbf{x}-\mathbf{x}_0)+(\mathbf{r}-\mathbf{r}_0),s-s_0\,|\,\mathbf{r}_0,\tau_0). \tag{3.5}$$

Let us now assume that the Reynolds number of the flow is sufficiently high that an inertial range of wave-numbers, or eddy sizes, exists. By this we mean that the wave-numbers which contain most of the energy are distinct from the higher wave-numbers which are responsible for most of the energy dissipation. Suppose now, that r_0 lies with this inertial range of eddy sizes. The eddies of dimension large compared to r_0 move the two marked particles together bodily without substantially altering the magnitude or direction of $\mathbf{r}_0$. In a frame of reference moving with these large-scale motions, the eddies of dimension small compared to r_0 are associated with a small r.m.s. velocity and have little effect upon $\mathbf{r}_0$. The rate of separation of the particles, in this case, is dominated by eddies of dimension $\sim r_0$, because such eddies make the principal contribution to the relative velocity of two points separated by a distance r_0 (cf. Batchelor 1953, ch. 6). These eddies disperse the particles substantially in a time of order

$$T(r_0) = r_0[v_0^2-U_{ii}(\mathbf{r}_0,0)]^{-\frac{1}{2}} \ll l/v_0. \tag{3.6}$$

For times short compared to $T(r_0)$, we may apply arguments similar to those of § 2. These show that (cf. equation (2.9))

$$B(\mathbf{x},t;\mathbf{y},s\,|\,\mathbf{x}_0,t_0;\mathbf{y}_0,s_0) = \frac{1}{(t-t_0)^3\,(s-s_0)^3}P\left(\frac{\mathbf{x}-\mathbf{x}_0}{t-t_0},\frac{\mathbf{y}-\mathbf{y}_0}{s-s_0}\right), \tag{3.7}$$

† A notational point must be noted here: the first argument of R in the definition (3.5) is $\mathbf{r}-\mathbf{r}_0$, the *change* in separation, and *not* $\mathbf{r}$, the separation itself.

where $P[\mathbf{u}_1(\mathbf{x}_0, t_0), \mathbf{u}_2(\mathbf{y}_0, s_0)]$ is the joint p.d.f. for velocity $\mathbf{u}_1$ at position $\mathbf{x}_0$ and time t_0 and velocity $\mathbf{u}_2$ at position $\mathbf{y}_0$ and time s_0. It follows (cf. equation (2.17)) that

$$\begin{aligned}\langle \Delta x_i \Delta y_j \rangle &= \iint \Delta x_i \Delta y_j B(\mathbf{x}, t; \mathbf{y}, s \mid \mathbf{x}_0, t_0; \mathbf{y}_0, s_0)\, d\mathbf{x}\, d\mathbf{y} \\ &= 2U_{ij}(\mathbf{x}_0, t_0; \mathbf{y}_0, s_0)\,(t-t_0)\,(s-s_0), \end{aligned} \tag{3.8}$$

where $\Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0$ and $\Delta \mathbf{y} = \mathbf{y} - \mathbf{y}_0$. Also, for these short times, (3.7) shows that Richardson's function is

$$R(\mathbf{r}, t, t \mid \mathbf{x}_0, \mathbf{r}_0, t_0, t_0) = \frac{1}{(t-t_0)^3} \mathscr{P}\left(\frac{\mathbf{r}-\mathbf{r}_0}{t-t_0}\right), \tag{3.9}$$

where $\mathscr{P}[\mathbf{V}(\mathbf{r}_0, t_0)]$ is the p.d.f. of *relative* velocity $\mathbf{V}$ between the two points $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{r}_0$ at time t_0. It follows from (3.9) that

$$\langle \Delta r_i \Delta r_j \rangle = \langle V_i V_j \rangle (t-t_0)^2, \tag{3.10}$$

which, for an isotropic field with r_3-axis along $\mathbf{r}_0$, gives

$$\left.\begin{aligned} \langle (\Delta r_1)^2 \rangle = \langle (\Delta r_2)^2 \rangle &= 2v_1^2[1 - g(r_0, t_0)]\,(t-t_0)^2, \\ \langle (\Delta r_3)^2 \rangle &= 2v_1^2[1 - f(r_0, t_0)]\,(t-t_0)^2, \end{aligned}\right\} \tag{3.11}$$

where $f(r,t)$ and $g(r,t) = f(r,t) + \frac{1}{2} r\, \partial f(r,t)/\partial r$ are respectively the longitudinal and transverse velocity correlations at time t for points separated by a distance r. Thus, Richardson's function is initially oblate spheroidal with the line joining the origin ($\mathbf{r} = 0$) to $\mathbf{r}_0$ as axis. All these results for short times are essentially due to Batchelor (1952*a*) and are included here for comparison purposes (see § 3.3).

3.2. *Integro-differential equations for* $B(\mathbf{x}, t; \mathbf{y}, s \mid \mathbf{x}_0, t_0; \mathbf{y}_0, s_0)$

When $t - t_0$ and $s - s_0$ are not small compared to $T(r_0)$, it is no longer legitimate to ignore the spatial and temporal variations of $u_i(\mathbf{x}, t)$. It is, nevertheless, possible to effect a partial summation of the formal solution for B containing terms of all orders in the expansion. Since the determination of B is a problem which is essentially inhomogeneous (even if the velocity field is homogeneous), the Fourier modes are not weakly dependent and the methods expounded by Kraichnan (1959) are not applicable. However, Kraichnan (1961) has recently generalized his methods to inhomogeneous problems, and has given an approximate equation of motion for the covariance $\langle \psi(\mathbf{x}, t)\, \psi(\mathbf{x}', t') \rangle$ of a convected passive scalar field. A straightforward generalization to our case of two scalar fields yields the result

$$\begin{aligned} &\frac{\partial}{\partial t} B(\mathbf{x}, t; \mathbf{y}, s \mid \mathbf{x}_0, t_0; \mathbf{y}_0, s_0) \\ &\quad = \int_{t_0}^{t} dt' \int d\mathbf{x}'\, U_{ij}(\mathbf{x}, t; \mathbf{x}', t') \frac{\partial G(\mathbf{x}, t \mid \mathbf{x}', t')}{\partial x_i} \frac{\partial B(\mathbf{x}', t'; \mathbf{y}, s \mid \mathbf{x}_0, t_0; \mathbf{y}_0, s_0)}{\partial x'_j} \\ &\qquad + \int_{s_0}^{s} ds' \int d\mathbf{y}'\, U_{ij}(\mathbf{x}, t; \mathbf{y}', s')\, G(\mathbf{y}, s \mid \mathbf{y}', s') \frac{\partial^2 B(\mathbf{x}, t; \mathbf{y}', s' \mid \mathbf{x}_0, t_0; \mathbf{y}_0, s_0)}{\partial x_i \partial y'_j}, \end{aligned} \tag{3.12}$$

where $G(\mathbf{x}, t \mid \mathbf{x}_0, t_0)$ is the one-point Green function of § 2. An equation similar to (3.12) holds for $\partial B/\partial s$.

The equation for $R(\mathbf{r},t,s\,|\,\mathbf{x}_0,\mathbf{r}_0,t_0,s_0)$ that can be obtained from (3.12) by integration may also, in the case of homogeneous velocity fields, be derived quite easily by a device which reduces the problem to a fully homogeneous one. Instead of imagining a single pair of fluid elements placed initially (i.e. at $t=t_0$, $s=s_0$) at $\mathbf{x}_0$ and $\mathbf{y}_0$ ($\mathbf{y}_0-\mathbf{x}_0=\mathbf{r}_0$), we imagine an infinite number of such pairs which are statistically independent and distributed homogeneously throughout space. All these pairs have the same separation $\mathbf{r}_0$, and for each pair the signs of ψ_1 and ψ_2 are, with equal probability, either both positive or both negative; the signs assigned to different pairs are statistically independent of each other. The ψ_1 and ψ_2 fields constructed in this way are strictly homogeneous, have zero means, and are such that

$$\langle\psi_1(\mathbf{x},t)\,\psi_2(\mathbf{x}+\mathbf{r},s)\rangle = R(\mathbf{r}-\mathbf{r}_0,t,s\,|\,\mathbf{r}_0,t_0,s_0), \tag{3.13}$$

where the left-hand side refers to the homogeneous problem just constructed and the right-hand side is the Richardson function for the original inhomogeneous problem.

Having established this correspondence, we again adopt cyclic boundary conditions and expand ψ_1, ψ_2 and $\mathbf{u}$ for the homogeneous problem in the forms

$$\psi_1(\mathbf{x},t) = \sum_{\mathbf{k}}\tilde{\psi}_1(\mathbf{k},t)\,e^{i\mathbf{k}.(\mathbf{x}-\mathbf{x}_0)}, \tag{3.14}$$

$$\psi_2(\mathbf{y},s) = \sum_{\mathbf{k}}\tilde{\psi}_2(\mathbf{k},s)\,e^{i\mathbf{k}.(\mathbf{y}-\mathbf{y}_0)}, \tag{3.15}$$

$$\mathbf{u}(\mathbf{x},t) = \sum_{\mathbf{k}}\tilde{\mathbf{u}}(\mathbf{k},t)\,e^{i\mathbf{k}.\mathbf{x}}, \tag{3.16}$$

(cf. K., equation (2.1) and footnote on p. 263 above). Then equation (2.2) for the ψ_1 field may be written

$$\frac{\partial\tilde{\psi}_1(\mathbf{k},t)}{\partial t} + ik_i\sum_{\mathbf{q}}\tilde{u}_i(\mathbf{p},t)\,e^{-i\mathbf{p}.\mathbf{x}_0}\,\tilde{\psi}_1(\mathbf{q},t) = 0, \tag{3.17}$$

and, for the ψ_2 field,

$$\frac{\partial\tilde{\psi}_2(\mathbf{k},s)}{\partial s} + ik_i\sum_{\mathbf{q}}\tilde{u}_i(\mathbf{p},s)\,e^{-i\mathbf{p}.\mathbf{y}_0}\,\tilde{\psi}_2(\mathbf{q},s) = 0, \tag{3.18}$$

where $\mathbf{p}=\mathbf{k}-\mathbf{q}$. By (3.17) we have (cf. K., equations (3.4) and (3.5))

$$\frac{\partial}{\partial t}\langle\tilde{\psi}_1(-\mathbf{k},t)\,\tilde{\psi}_2(\mathbf{k},s)\rangle = ik_i\sum_{\mathbf{q}}\langle\tilde{u}_i(-\mathbf{p},t)\,\tilde{\psi}_1(-\mathbf{q},t)\,\tilde{\psi}_2(\mathbf{k},s)\rangle\,e^{i\mathbf{p}.\mathbf{x}_0}. \tag{3.19}$$

Now, in the limit $L\to\infty$, the direct interaction approximation (K., § 2.4) gives

$$\begin{aligned}\langle\tilde{u}_i(-\mathbf{p},t)\,\tilde{\psi}_1(-\mathbf{q},t)\,\tilde{\psi}_2(\mathbf{k},s)\rangle \\ = \langle\tilde{u}_i(-\mathbf{p},t)\,\tilde{\psi}_{1,-\mathbf{k}}(-\mathbf{q},t)\,\tilde{\psi}_2(\mathbf{k},s)\rangle + \langle\tilde{u}_i(-\mathbf{p},t)\,\tilde{\psi}_1(-\mathbf{q},t)\,\tilde{\psi}_{2,\mathbf{q}}(\mathbf{k},s)\rangle,\end{aligned} \tag{3.20}$$

where (cf. equation (2.35) and K., § 3)

$$\tilde{\psi}_{1,-\mathbf{k}}(-\mathbf{q},t) = \int_{t_0}^{t} iq_j\tilde{u}_j(\mathbf{p},t')\,\tilde{\psi}_1(-\mathbf{k},t')\,\tilde{g}(\mathbf{q},t-t')\,e^{-i\mathbf{p}.\mathbf{x}_0}\,dt', \tag{3.21}$$

$$\tilde{\psi}_{2,\mathbf{q}}(\mathbf{k},s) = -\int_{s_0}^{s} ik_j\tilde{u}_j(\mathbf{p},s')\,\tilde{\psi}_2(\mathbf{q},s')\,\tilde{g}(\mathbf{k},s-s')\,e^{-i\mathbf{p}.\mathbf{y}_0}\,ds'. \tag{3.22}$$

Thus, applying the weak statistical dependence principle (K., § 2.2), we find that, in the limit $L \to \infty$ (cf. equations (2.38) to (2.40)),

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{R}(\mathbf{k},t,s\,|\,\mathbf{r}_0,\tau_0) &= -(2\pi)^3 k_i k_j \int_0^t dt' \int d\mathbf{q}\,\tilde{U}_{ij}(\mathbf{p},t')\,\tilde{G}(\mathbf{q},t')\,\tilde{R}(\mathbf{k},t-t',s\,|\,\mathbf{r}_0,\tau_0)\\ &\quad + (2\pi)^3 k_i k_j \int_0^s ds' \int d\mathbf{q}\,\tilde{U}_{ij}(\mathbf{p},\tau_0-t+s-s')\,e^{i\mathbf{p}\cdot\mathbf{r}_0}\,\tilde{G}(\mathbf{k},s')\,\tilde{R}(\mathbf{q},t,s-s'\,|\,\mathbf{r}_0,\tau_0).\end{aligned} \tag{3.23}$$

Returning to $\mathbf{x}$-space by the arguments of (2.41) to (2.43) this becomes

$$\begin{aligned}\frac{\partial R(\mathbf{r},t,s\,|\,\mathbf{r}_0,\tau_0)}{\partial t} &= \frac{\partial^2}{\partial r_i \partial r_j}\Big[\int_0^t dt' \int d\mathbf{x}'\,U_{ij}(\mathbf{x}',t')\,G(\mathbf{x}',t')\,R(\mathbf{r}+\mathbf{x}',t-t',s\,|\,\mathbf{r}_0,\tau_0)\\ &\quad - \int_0^s ds' \int d\mathbf{y}'\,U_{ij}(\mathbf{r}_0+\mathbf{r}-\mathbf{y}',\tau_0+\tau-s')\,G(\mathbf{y}',s')\,R(\mathbf{r}-\mathbf{y}',t,s-s'\,|\,\mathbf{r}_0,\tau_0)\Big].\end{aligned} \tag{3.24}$$

This equation may also be derived directly from (3.12) which, when the velocity field is stationary and homogeneous, may be written after a partial integration in the form

$$\begin{aligned}\frac{\partial B(\mathbf{x},t;\mathbf{y},s\,|\,\mathbf{r}_0,\tau_0)}{\partial t} &= \frac{\partial^2}{\partial x_i \partial x_j}\int_0^t dt' \int d\mathbf{x}'\,U_{ij}(\mathbf{x}',t')\,G(\mathbf{x}',t')\,B(\mathbf{x}-\mathbf{x}',t-t';\mathbf{y},s\,|\,\mathbf{r}_0,\tau_0)\\ &\quad + \frac{\partial^2}{\partial x_i \partial y_j}\int_0^s ds' \int d\mathbf{y}'\,U_{ij}(\mathbf{r}_0-\mathbf{x}+\mathbf{y}-\mathbf{y}',\tau_0-t+s-s')\\ &\quad \times G(\mathbf{y}',s')\,B(\mathbf{x},t;\mathbf{y}-\mathbf{y}',s-s'\,|\,\mathbf{r}_0,\tau_0).\end{aligned} \tag{3.25}$$

The results (3.24) and (3.25)—consequences of Kraichnan's random coupling approximation—are the central results of this section.

We will now investigate some elementary consequences of equations (3.24) and (3.25). On multiplying each side of (3.24) by $r_i r_j$ and integrating over all $\mathbf{r}$ and the right-hand side by parts, we find

$$\begin{aligned}\frac{\partial}{\partial t}\int R(\mathbf{r},t,s\,|\,\mathbf{r}_0,\tau_0)\,r_i r_j\,d\mathbf{r} &= 2\int_0^t dt' \int d\mathbf{x}'\,U_{ij}(\mathbf{x}',t')\,G(\mathbf{x}',t')\\ &\quad - 2\int_0^s ds' \int d\mathbf{r}\,U_{ij}(\mathbf{r}_0+\mathbf{r},\tau_0+s'-t)\,R(\mathbf{r},t,s'\,|\,\mathbf{r}_0,\tau_0).\end{aligned} \tag{3.26}$$

By differentiating (3.26) with respect to s, we find

$$\frac{\partial^2}{\partial t\,\partial s}\int R(\mathbf{r},t,s\,|\,\mathbf{r}_0,\tau_0)\,r_i r_j\,d\mathbf{r} = -2\int d\mathbf{r}\,U_{ij}(\mathbf{r}_0+\mathbf{r},\tau_0+\tau)\,R(\mathbf{r},t,s\,|\,\mathbf{r}_0,\tau_0). \tag{3.27}$$

Let us take $t_0 = 0$, $s_0 = 0$. Then, by (3.27) and (2.50), we see that the Lagrangian correlation for relative velocity $\mathbf{V}$ between two particles which were initially separated by a distance $\mathbf{r}_0$ is

$$\langle V_i(t)\,V_j(s)\rangle_L = 2\int d\mathbf{r}\,U_{ij}(\mathbf{r},s-t)\,[G(\mathbf{r},s-t) - R(\mathbf{r}-\mathbf{r}_0,t,s\,|\,\mathbf{r}_0,0)]. \tag{3.28}$$

This result should be compared to the exact result

$$\langle V_i(t)\,V_j(s)\rangle_L = \iint d\mathbf{x}\,d\mathbf{y}[\langle\psi_1(\mathbf{x},t)\,\psi_1(\mathbf{y},s)\,u_i(\mathbf{x},t)\,u_j(\mathbf{y},s)\rangle + \langle\psi_2(\mathbf{x},t)\,\psi_2(\mathbf{y},s)\,u_i(\mathbf{x},t)\,u_j(\mathbf{y},s)\rangle - \langle\psi_1(\mathbf{x},t)\,\psi_2(\mathbf{y},s)\,u_i(\mathbf{x},t)\,u_j(\mathbf{y},s)\rangle - \langle\psi_2(\mathbf{x},t)\,\psi_1(\mathbf{y},s)\,u_i(\mathbf{x},t)\,u_j(\mathbf{y},s)\rangle], \tag{3.29}$$

where $\psi_1(\mathbf{x},t) = \psi_1(\mathbf{x},t\,|\,\mathbf{x}_0,0)$ and $\psi_2(\mathbf{y},s) = \psi_2(\mathbf{y},s\,|\,\mathbf{y}_0,0)$ denote the unaveraged Green functions. In stationary homogeneous flows, this result reduces to (3.28) on the assumption that the ψ and $\mathbf{u}$ fields are statistically independent (cf. § 2.3).

Also, by adding to (3.26) the analogous equation for the derivative with respect to s, and setting $t = s$, we find

$$\begin{aligned}\frac{\partial}{\partial t}\langle r_i r_j\rangle &= \int\frac{\partial}{\partial t}R(\mathbf{r},t,t\,|\,\mathbf{r}_0,0)\,r_i r_j\,d\mathbf{r}\\ &= 4\int_0^t dt'\int d\mathbf{x}'\,U_{ij}(\mathbf{x}',t')\,G(\mathbf{x}',t')\\ &\quad -2\int_0^t dt'\int d\mathbf{r}\,U_{ij}(\mathbf{r}_0+\mathbf{r},t-t')\,[R(\mathbf{r},t,t'\,|\,\mathbf{r}_0,0)+R(\mathbf{r},t',t\,|\,\mathbf{r}_0,0)].\end{aligned} \tag{3.30}$$

For $t \ll T(r_0)$, the particles have not had time to change their separation greatly and R is negligible except near $\mathbf{r} = 0$. Thus the right-hand side of (3.30) becomes $4[U_{ij}(0,\,0) - U_{ij}(\mathbf{r}_0,0)]\,t$, in agreement with (3.10). For $t \gg l/v_0$, R is appreciable even at separations r of order l. For such large separations, the second term on the right-hand side of (3.30) is quite negligible, and we find (cf. equations (2.44) and (2.45))

$$\langle r_i r_j\rangle = 2\langle x_i x_j\rangle = 2\langle y_i y_j\rangle = 4t\kappa_{ij}. \tag{3.31}$$

This is consistent with the intuitive notion that, at such large separations from their source and each other, the particles will wander independently. For intermediate ranges of t, it appears that no such definite statements can be made. However, a reasonable approximation appears to be possible and this is discussed in § 3.4.

3.3 *Solution of the equations for short times*

For $t \ll T(r_0)$, we may assume that the G and B factors in the integrands of (3.25) are both negligible unless $\mathbf{x}', \mathbf{y}', \mathbf{x}, \mathbf{y}$ are small compared with $\mathbf{r}_0$. Then, the U_{ij} factors in the integrands may be replaced by their values for zero $\mathbf{x}', \mathbf{y}', \mathbf{x}, \mathbf{y}$ and similarly for the time arguments. Therefore, in isotropic flows, we have

$$\begin{aligned}\frac{\partial B(\mathbf{x},t;\mathbf{y},s\,|\,\mathbf{r}_0,\tau_0)}{\partial t} &= v_1^2\delta_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\int_0^t dt'\int d\mathbf{x}'\,G(\mathbf{x}',t')\,B(\mathbf{x}-\mathbf{x}',t-t';\mathbf{y},s\,|\,\mathbf{r}_0,\tau_0)\\ &\quad - U_{ij}(\mathbf{r}_0,\tau_0)\frac{\partial^2}{\partial x_i\partial y_j}\int_0^s ds'\int d\mathbf{y}'G(\mathbf{y}',s')\,B(\mathbf{x},t;\mathbf{y}-\mathbf{y}',s-s'\,|\,\mathbf{r}_0,\tau_0).\end{aligned} \tag{3.32}$$

Take a combined Laplace and Fourier transform defined by

$$\tilde{B}^*(\mathbf{k},p;\mathbf{l},q\,|\,\mathbf{r}_0,\tau_0) = \frac{1}{(2\pi)^6}\int_0^\infty dt\,e^{-pt}\int_0^\infty ds\,e^{-qs}\int d\mathbf{x}\,e^{i\mathbf{k}.\mathbf{x}}\int d\mathbf{y}\,e^{i\mathbf{l}.\mathbf{y}}B(\mathbf{x},t;\mathbf{y},s\,|\,\mathbf{r}_0,\tau_0). \tag{3.33}$$

Then, by (3.32),

$$p\tilde{B}^*(\mathbf{k},p;\mathbf{l},q\,|\,\mathbf{r}_0,\tau_0) - \frac{1}{(2\pi)^3}\tilde{G}^*(\mathbf{l},q)$$
$$= -(2\pi)^3 k^2v_1^2\tilde{G}^*(\mathbf{k},p)\,\tilde{B}^*(\mathbf{k},p;\mathbf{l},q\,|\,\mathbf{r}_0,\tau_0)$$
$$-(2\pi)^3 U_{ij}(\mathbf{r}_0,\tau_0)\,k_i l_j\tilde{G}^*(\mathbf{l},q)\,\tilde{B}^*(\mathbf{k},p;\mathbf{l},q\,|\,\mathbf{r}_0,\tau_0), \tag{3.34}$$

where $\tilde{G}^*(\mathbf{k},p)$ is the Fourier transform with respect to $\mathbf{x}$ and the Laplace transform with respect to t of $G(\mathbf{x},t)$. By (2.57), we have

$$p + (2\pi)^3 k^2v_1^2\tilde{G}^*(\mathbf{k},p) = \frac{1}{(2\pi)^3\tilde{G}^*(\mathbf{k},p)}. \tag{3.35}$$

By (3.34) and (3.35), it follows that

$$\tilde{B}^*(\mathbf{k},p;\mathbf{l},q\,|\,\mathbf{r}_0,\tau_0) = \frac{\tilde{G}^*(\mathbf{k},p)\,\tilde{G}^*(\mathbf{l},q)}{1+(2\pi)^6 U_{ij}(\mathbf{r}_0,\tau_0)\,k_i l_j\tilde{G}^*(\mathbf{k},p)\,\tilde{G}^*(\mathbf{l},s)}. \tag{3.36}$$

If we expand the denominator of (3.36), we obtain

$$\tilde{B}^*(\mathbf{k},p;\mathbf{l},q\,|\,\mathbf{r}_0,\tau_0) = \sum_0^\infty(-1)^m[(2\pi)^6 U_{ij}(\mathbf{r}_0,\tau_0)\,k_i l_j]^m\,[\tilde{G}^*(\mathbf{k},p)\,\tilde{G}^*(\mathbf{l},q)]^{m+1}. \tag{3.37}$$

Now, by (3.35) $$\tilde{G}^*(\mathbf{k},p) = \frac{1}{4\pi^3[p+\sqrt{(p^2+4k^2v_1^2)}]}. \tag{3.38}$$

The inverse Laplace transformation of $[\tilde{G}^*(\mathbf{k},p)]^{m+1}$ is therefore (see, for example, Watson 1944, §13.2, equation (7))

$$\frac{(m+1)\,J_{m+1}(2kv_1t)}{(2\pi)^{3m+3}\,(kv_1)^{m+1}t}.$$

Thus

$$\tilde{B}(\mathbf{k},t;\mathbf{l},s\,|\,\mathbf{r}_0,\tau_0) = \frac{1}{(2\pi)^6}\sum_0^\infty(-1)^m\,(m+1)^2\left[\frac{k_il_j\,U_{ij}(r_0,\tau_0)}{klv_1^2}\right]^m \times\frac{J_{m+1}(2kv_1t)}{kv_1t}\,\frac{J_{m+1}(2lv_is)}{lv_1s}. \tag{3.39}$$

The second moments of (3.39) agree with (3.8); the higher moments are given with progressively less accuracy (cf. §2.4). Roberts (1960) has discussed the significance of (3.39) in terms of a diagram expansion. The relationship

$$\int\frac{J_{m+1}(2kv_1t)}{(kv_1t)^{m+1}}\,e^{i\mathbf{k}.\mathbf{x}}d\mathbf{x} = \begin{cases}\dfrac{2\pi m!}{(2m)!}\,\dfrac{(4v_1^2t^2-x^2)^{m-\frac{1}{2}}}{(v_1t)^{2m+2}} & \text{if}\quad x<2v_1t,\\[2ex] 0 & \text{if}\quad x>2v_1t,\end{cases}$$

can be obtained by induction from results of Watson (1944, § 13.42, equation (4), $\mu = 1$, using also § 13.24, equation (1)) or directly (Watson, § 13.14). This enables us to invert the function $\tilde{B}(\mathbf{k}, t; \mathbf{l}, s \mid \mathbf{r}_0, \tau_0)$ in the form (cf. equation (2.59))

$$B(\mathbf{x}, t; \mathbf{y}, s \mid \mathbf{r}_0, \tau_0) = \begin{cases} \dfrac{1}{(2\pi)^4 (v_1 t)^3 (v_1 s)^3} \sum_0^\infty (-1)^m \dfrac{[(m+1)!]^2}{[(2m)!]^2} \left[U_{ij}(\mathbf{r}_0, \tau_0) \dfrac{\partial^2}{\partial x_i \partial y_j} \right]^m & \\ \qquad \times \left[\left(1 - \dfrac{x^2}{4v_1^2 t^2}\right) \left(1 - \dfrac{y^2}{4v_1^2 s^2}\right) \right]^{m-\frac{1}{2}} & \text{if} \quad x < 2v_1 t \text{ and } y < 2v_1 s, \\ 0 & \text{if} \quad x > 2v_1 t \text{ or } y > 2v_1 s. \end{cases} \tag{3.40}$$

This result shows consistency with the results for one-particle diffusion as in § 2.3, the effect of the artificial cut-off $2v_1$ in the p.d.f. of velocity is apparent. It is also clear that, if $U_{ij}(\mathbf{r}_0, \tau_0)$ is small compared to v_1^2, (3.40) reduces to its first term

$$B(\mathbf{x}, t; \mathbf{y}, s \mid \mathbf{r}_0, \tau_0) \doteqdot G(\mathbf{x}, t)\, G(\mathbf{y}, s),$$

where the right-hand side is given by (2.59). If the initial separation of the points is so close in space and time that we can write $U_{ij}(\mathbf{r}_0, \tau_0) = v_1^2 \delta_{ij}$, equation (3.40) shows (cf. Watson, § 11.41, equation (12)) that

$$\tilde{R}(\mathbf{k}, t, s \mid \mathbf{r}_0, \tau_0) = (2\pi)^3 \tilde{B}(-\mathbf{k}, t; \mathbf{k}, s \mid \mathbf{r}_0, \tau_0) = \frac{1}{(2\pi)^3} \frac{J_1[2kv_1(s-t)]}{kv_1(s-t)},$$

or, inverting,

$$R(\mathbf{r}, t, s \mid \mathbf{r}_0, \tau_0) \to G(\mathbf{r}_0 + \mathbf{r}, s-t); \quad \mathbf{r}_0 \to 0, \quad \tau_0 \to 0. \tag{3.41}$$

This shows, as we expect, that in the limit $\mathbf{r}_0 \to 0$, $\tau_0 \to 0$, the particles are not separated by the flow and simply move together as one particle. (As a further consistency check, we note that the Lagrangian correlation (3.28) is given correctly by (3.41) in this case.)

3.4. *Solutions for large and intermediate times*

For very long times $t \gg l/v_0$, there is a high probability that the particles have separated by a distance comparable to, or greater than, l and therefore wander independently. In fact, the solution of (3.25) is exactly analogous to that of § 2.5 for the one-point Green function, and we find

$$B(\mathbf{x}, t; \mathbf{y}, s \mid \mathbf{r}_0, \tau_0) \to G(\mathbf{x}, t)\, G(\mathbf{y}, s) \quad (r \gg l), \tag{3.42}$$

where $G(\mathbf{x}, t)$ and $G(\mathbf{y}, s)$ are given, in homogeneous flows, by the appropriate solutions of (2.69). In this case we find $R(\mathbf{r}, t)$ satisfies

$$\frac{\partial R(\mathbf{r}, t)}{\partial t} = 2\kappa_{ij}^\infty \frac{\partial^2 R(\mathbf{r}, t)}{\partial r_i \partial r_j},$$

so that $R(\mathbf{r}, t)$ assumes a Gaussian form corresponding to a diffusivity twice that characteristic of the one particle Green function (cf. equations (2.69) and (3.31)).

Consider turbulence at the high Reynolds numbers. Since the kinetic energy density of the turbulent flow must be finite, the spectrum function $E(k)$ must satisfy

$$kE(k) \to 0, \quad k \to \infty; \quad kE(k) \to 0, \quad k \to 0.$$

Also $\int_0^k E(k)\,dk \doteqdot \frac{1}{2}v_0^2$ if $k \gg k_0$, $\int_k^\infty E(k)\,k^2\,dk \doteqdot \epsilon/2\nu$ if $k \ll k_d$,

where $k_0\,(=1/l)$ and k_d are wave-numbers characteristic of the energy-containing range and the dissipation range, respectively. (ϵ = rate of dissipation of energy per unit mass; ν = kinematical viscosity.) Thus, within the inertial range $k_0 \ll k \ll k_d$, the spectrum must be such that

$$\int_{k_0}^{k_d} E(k)\,dk \ll \tfrac{1}{2}v_0^2 \quad \text{and} \quad \int_{k_0}^{k_d} E(k)\,k^2\,dk \ll \epsilon/2\nu.$$

It follows that, if the inertial range spectrum approximates to a power law, it must be such that

$$kE(k) \to 0, \quad k \to \infty; \quad k^3E(k) \to 0, \quad k \to 0.$$

For the later developments (cf. equations (3.61) and (3.62) below), we will require the more stringent conditions

$$kE(k) \to 0, \quad k \to \infty; \quad k^2E(k) \to 0, \quad k \to 0. \tag{3.43}$$

If the initial separation satisfies

$$l \gg r_0 \gg 1/k_d, \tag{3.44}$$

we expect that, in a time large compared to $T(r_0)$ but small compared to l/v_0, $\langle(\mathbf{r}-\mathbf{r}_0)^2\rangle$ will become large compared to r_0^2 but remain small compared to l^2. For these 'intermediate times' (as we shall term them), neither the short-time solution of § 3.3 nor the long-time solution above is valid.

Consider $R(\mathbf{r}-\mathbf{r}_0, t, t+\tau \mid \mathbf{r}_0, 0)$ for $\tau > 0$. This quantity is the p.d.f. of the separation $\mathbf{r}$ of two particles (released at $t = 0$ at a separation of $\mathbf{r}_0$) one of which has been carried by the flow for a time t, and the other for a time $t+\tau$. This process may be visualized in two stages. During the first, of duration t, both particles are carried by the flow. Their separation $\mathbf{r}$ during this time is essentially unaffected by the energy-containing eddies which give nearly equal displacements to both particles. It is governed by the motions (relative to the energy-containing eddies) of dimension $\sim r$. Now, in a frame moving with the energy-containing eddies, the r.m.s. velocity associated with these small-scale motions is very small compared to v_0. Thus, the mean square separation $\langle r^2\rangle$ at the termination of the first stage is very small compared to $(v_0t)^2$. During the second stage, of duration τ, one of the particles can be considered as fixed in space while the other is carried by the flow for a further time τ. Its motion during this time is dominated by the energy-containing motions. During the first stage the relative diffusion is given by $R(\mathbf{r}, t, t \mid \mathbf{r}_0, 0)$. During the second stage the further diffusion of the second particle should be given by $G(\mathbf{r}, \tau)$, since the energy-containing motions are almost uncorrelated with the small-scale motions. Thus, we expect

$$R(\mathbf{r}, t, t+\tau \mid \mathbf{r}_0, 0) \doteqdot \int R(\mathbf{r}', t, t \mid \mathbf{r}_0, 0)\, G(\mathbf{r}-\mathbf{r}', \tau)\, d\mathbf{r}'. \tag{3.45}$$

It is clear that this result is exact for $t = 0$ or $\tau = 0$. Also, the change in the mean square separation during the second stage will be of the order of $(v_0\tau)^2$. Thus, when this is large compared to the value of $\langle r^2 \rangle$ at the end of the first stage, (3.45) becomes

$$R(\mathbf{r}, t, t+\tau \mid \mathbf{r}_0, 0) \doteqdot G(\mathbf{r}, \tau). \tag{3.46}$$

Consider now (3.24) and its counterpart for $\partial R/\partial s$. These show that

$$\frac{\partial R(\mathbf{r}, t, t \mid \mathbf{r}_0, 0)}{\partial t} = \frac{\partial^2}{\partial r_i \partial r_j}\left\{\int_0^t dt' \int d\mathbf{r}'[U_{ij}(\mathbf{r}', t') - U_{ij}(\mathbf{r}_0 + \mathbf{r} - \mathbf{r}', t')] \times G(\mathbf{r}', t')[R(\mathbf{r}-\mathbf{r}', t-t', t \mid \mathbf{r}_0, 0) + R(\mathbf{r}-\mathbf{r}', t, t-t' \mid \mathbf{r}_0, 0)]\right\}. \tag{3.47}$$

By the arguments above, the integrand should be dominated by contributions from small t' (because of the behaviour of R) and from small $\mathbf{r}'$ (because of the behaviour of G). We will therefore replace the R factors in the integrand by $R(\mathbf{r}, t, t \mid \mathbf{r}_0, 0)$. It seems likely from the preceding discussion that this approximation will lead to results which are at least qualitatively correct.

We now find

$$\frac{\partial R(\mathbf{r}, t, t \mid \mathbf{r}_0, 0)}{\partial t} = \frac{\partial^2}{\partial r_i \partial r_j}[K_{ij}(\mathbf{r}, t) R(\mathbf{r}, t, t \mid \mathbf{r}_0, 0)], \tag{3.48}$$

where
$$K_{ij}(\mathbf{r}, t) = 2\int_0^t dt' \int d\mathbf{r}'[U_{ij}(\mathbf{r}', t') - U_{ij}(\mathbf{r}_0 + \mathbf{r} - \mathbf{r}', t')]\, G(\mathbf{r}', t'). \tag{3.49}$$

For short times $t \ll T(r_0)$, (3.49) becomes

$$K_{ij}(\mathbf{r}, t) = 2t[v_1^2 \delta_{ij} - U_{ij}(\mathbf{r}_0, 0)], \tag{3.50}$$

which, by (3.48), is in agreement with (3.10). For very large times $t \gg l/v_0$, there is a high probability that the separation of particles is $\gtrsim l$. For these separations, the second U factor in the integrand of (3.49) is negligible, and the first factor gives

$$K_{ij}(\mathbf{r}, t) = 2\kappa_{ij}, \tag{3.51}$$

in agreement with the results derived earlier (cf. equation (3.31)).

To calculate the form of $K_{ij}(\mathbf{r}, t)$ for intermediate times, we express (3.49) in the form

$$K_{ij}(\mathbf{r}, t) = 2(2\pi)^3 \int_0^t dt' \int d\mathbf{k}\, [1 - e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{r}_0)}]\, \tilde{U}_{ij}(\mathbf{k}, t')\, \tilde{G}(\mathbf{k}, t'). \tag{3.52}$$

Assuming isotropy, we can write

$$\tilde{U}_{ij}(\mathbf{k}, t) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \frac{E(\mathbf{k}, t)}{4\pi k^2}, \tag{3.53}$$

where $E(k, 0)$ is the energy spectrum. It follows that if we adopt spherical polar co-ordinates and write

$$R(\mathbf{r} - \mathbf{r}_0, t, t \mid \mathbf{r}_0, 0) = R(r, \theta, \phi, t),$$

(3.48) has the form

$$\frac{\partial R}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 K(r, t)\frac{\partial R}{\partial r}\right] + \frac{1}{r^2} M(r, t)\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial R}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 R}{\partial \phi^2}\right], \tag{3.54}$$

where, by (3.52) and (3.53),

$$K(r, t) = 4(2\pi)^3 \int_0^t dt' \int_0^\infty dk \left[\frac{1}{3} - \frac{\sin kr}{(kr)^3} + \frac{\cos kr}{(kr)^2}\right] E(k, t')\, \tilde{G}(k, t'), \tag{3.55}$$

$$M(r, t) = 2(2\pi)^3 \int_0^t dt' \int_0^\infty dk \left[\frac{2}{3} - \frac{\sin kr}{kr} + \frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2}\right] E(k, t')\, \tilde{G}(k, t'). \tag{3.56}$$

For small kr, the quantities in the square brackets [......] in the integrands of (3.55) and (3.56) are proportional to $(kr)^2$. Also, in the inertial range, we may take (cf. equation (2.21))

$$\tilde{G}(k,t) = \frac{1}{(2\pi)^3}\exp\left[-\tfrac{1}{2}k^2v_1^2t^2\right], \tag{3.57}$$

and (cf. K., §8.4) $$E(k,t) = E(k)\exp\left[-\tfrac{1}{2}k^2v_1^2t^2\right]. \tag{3.58}$$

Thus, since by our initial supposition (3.43), $k^2E(k) \to 0$, $k \to 0$, it follows that the integrands of (3.55) and (3.56) do not tend to infinity as rapidly as k^{-1}, $k \to 0$. Consequently the form of $E(k)$ for small k ($\sim k_0$) does not influence the values of $K(r,t)$ and $M(r,t)$ appreciably. However, these quantities do depend implicitly on the energy-containing range through the forms (3.57) and (3.58) for $\tilde{G}(k,t)$ and $E(k,t)$. We will discuss this in more detail at the conclusion of this section.

Two further approximations are clearly justified. First, since $\langle(\mathbf{r}-\mathbf{r}_0)^2\rangle$ is large compared to r_0^2 for the times under consideration, the particles have in this time lost all 'memory' of their initial separation $\mathbf{r}_0$. Thus $R(r,\theta,\phi,t)$ must be independent of r_0, θ, and ϕ, and, by equation (3.54), it must satisfy

$$\frac{\partial R(r,t)}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2K(r,t)\frac{\partial R(r,t)}{\partial r}\right]. \tag{3.59}$$

Secondly, since $\langle r^2\rangle$ is small compared to $(v_0t)^2$, the exponential factors of (3.57) and (3.58) are small at the upper limit of integration over t in (3.55) and (3.56), and we may therefore write

$$\begin{aligned} K(r,t) = K(r) &= 4\int_0^\infty dt\int_0^\infty dk\left[\frac{1}{3}-\frac{\sin kr}{(kr)^3}+\frac{\cos kr}{(kr)^2}\right]E(k)\exp\left[-\tfrac{1}{3}k^2v_0^2t^2\right] \\ &= 2\sqrt{3\pi}\int_0^\infty \frac{dk}{v_0k}\left[\frac{1}{3}-\frac{\sin kr}{(kr)^3}+\frac{\cos kr}{(kr)^2}\right]E(k). \end{aligned} \tag{3.60}$$

Equation (3.59) was proposed by Richardson (1926) for the intermediate times discussed here. To investigate further the form of the variable diffusion coefficient (3.60) we will assume that in the inertial range the spectrum is a power law† of the form (cf. equation (3.43))

$$E(k) = \beta\epsilon^{n-1}v_0^{5-3n}k^{-n} \quad (1 < n < 2), \tag{3.61}$$

† We may now take the wave-number k_d characteristic of the dissipation range to be

$$k_d = R_0^{1/(3-n)}k_0,$$

where $$k_0 \equiv 1/l = \epsilon/v_0^3, \quad R_0 = v_0^4/\epsilon\nu = v_0/k_0\nu.$$

The 'intermediate times' referred to above may now be defined more precisely by

$$T(r_0) = \left(\frac{r_0}{l}\right)^{\frac{1}{2}(3-n)}\frac{l}{v_0} \ll t \ll \frac{l}{v_0}.$$

Also, the circumstances under which (3.46) is a good approximation are

$$v_0\tau/l \gg (v_0t/l)^{1/(2-n)}.$$

Note that (3.61) may be written

$$E(k) = \beta lv_0^2(kl)^{-n}.$$

where β is a dimensionless constant of the order of unity. With this spectrum, (3.60) gives, on integrating by parts and use of results of Watson (1944, §§3.4, 13.24),

$$K(r) = \frac{\beta\pi\sqrt{6}}{n}\epsilon^{n-1}v_0^{\frac{4}{3}-3n}r^n\int_0^\infty \frac{J_{\frac{5}{2}}(x)\,dx}{x^{\frac{1}{2}(n+3)}}$$

$$= \frac{\beta\pi\sqrt{3}}{2^{n+1}n}\frac{\Gamma(1-\frac{1}{2}n)}{\Gamma(\frac{5}{2}+\frac{1}{2}n)}\epsilon^{n-1}v_0^{\frac{4}{3}-3n}r^n \equiv \lambda r^n, \text{ (say)}. \tag{3.62}$$

If we suppose that at some time $t_1 \gg T(r_0)$

$$R(r,t) = \mathscr{R}(r,t_1), \tag{3.63}$$

then at subsequent times $t(\ll l/v_0)$, (3.59) and (3.62) show that

$$R(r,t) = \tfrac{1}{2}(2-n)\,r^{-\frac{1}{2}(n+1)}\int_0^\infty \tilde{\mathscr{R}}(\xi,t_1)\,J_s(\xi r^{\frac{1}{2}(2-n)})\exp\left[-\tfrac{1}{4}\lambda(2-n)^2\xi^2(t-t_1)\right]\xi\,d\xi, \tag{3.64}$$

where $s = (1+n)/(2-n)$ and

$$\tilde{\mathscr{R}}(\xi,t_1) = \int_0^\infty r^{\frac{1}{2}(3-n)}\mathscr{R}(r,t_1)\,J_s(\xi r^{\frac{1}{2}(2-n)})\,dr. \tag{3.65}$$

For $t \gg t_1$, (3.64) reduces to†

$$R(r,t) = \frac{1}{(2-n)^{(4+n)/(2-n)}\,\Gamma\left(\dfrac{3}{2-n}\right)}\cdot\frac{1}{(\lambda t)^{3/(2-n)}}\exp\left[-r^{2-n}/(2-n)^2\lambda t\right], \tag{3.66}$$

so that

$$\langle r^2\rangle = (2-n)^{4/(2-n)}\frac{\Gamma\left(\dfrac{5}{2-n}\right)}{\Gamma\left(\dfrac{3}{2-n}\right)}(\lambda t)^{2/(2-n)}. \tag{3.67}$$

The behaviour of $\langle r^2\rangle$ as a function of t is extraordinarily sensitive to the value of n assumed. Two cases are worthy of notice:

$n = \frac{3}{2}$ (Kraichnan 1959)

$$K(r) \propto \epsilon^{\frac{1}{2}}r^{\frac{3}{2}}/v_0^{\frac{1}{2}}, \quad \langle r^2\rangle \propto \epsilon^2 t^4/v_0^2, \tag{3.68}$$

$n = \frac{5}{3}$ (Kolmogorov 1941)

$$K(r) \propto \epsilon^{\frac{2}{3}}r^{\frac{5}{3}}/v_0, \quad \langle r^2\rangle \propto \epsilon^4 t^6/v_0^6. \tag{3.69}$$

Neither of these agree with the form proposed by Richardson (1926, see also Batchelor 1950);

$$K(r) \propto \epsilon^{\frac{1}{3}}r^{\frac{4}{3}}, \quad \langle r^2\rangle \propto \epsilon t^3. \tag{3.70}$$

That this is so is not surprising. Kraichnan's direct interaction approximation does not give an inertial range spectrum which agrees with that derived from

† The behaviour of $R(r,t)$ and $\langle r^2\rangle$ given by (3.66) and (3.67) is almost certainly independent of the approximation

$$R(\mathbf{r}-\mathbf{r}', t-t', t \mid \mathbf{r}_0, 0) \doteqdot R(\mathbf{r}, t, t \mid \mathbf{r}_0, 0)$$

which led from (3.47) to (3.48). In fact, it can be shown that the more accurate approximation method based on (3.45) and (3.47) leads to results whose dimensional forms are identical to (3.66) and (3.67).

Kolmogorov's similarity arguments. In the same way, when this approximation is applied to turbulent diffusion, it does not give a diffusion coefficient which agrees with that derived by these similarity arguments, even if Kolmogorov's spectrum is assumed. This is because the dynamics of diffusion with a given velocity field differ on the two theories. It would seem that the behaviour of $\langle r^2 \rangle$ as a function of t might provide a sensitive test by which to confront with experiment different assumptions about the structure of the inertial range.

Kraichnan (1959) has made a detailed comparison between the direct-interaction approximation and the Kolmogorov theory. He has traced the difference in the inertial range spectra to the difference in the role played by the energy-containing eddies in the two cases. In Kolmogorov's theory these eddies merely convect the small-scale motions without influencing their dynamics, whereas in the direct-interaction approximation this is not so. In the same way, on arguments of the Kolmogorov type, the relative diffusion of particles should be independent of the energy-containing eddies. However, the application of the direct-interaction approximation has led to results which depend on the energy-containing motions. If we wish to modify our formalism in such a way that our results depend only on the small-scale motions, we would transform to a frame of reference moving with the energy-containing eddies. We would expect that results of the form (3.48), (3.49) would be qualitatively correct, provided that $\tilde{G}(k,t)$ now described one-point diffusion relative to a source moving with the energy-containing eddies. On similarity arguments of the Kolmogorov type, $\tilde{G}(k,t)$ would then depend upon $k^{\frac{2}{3}}\epsilon^{\frac{1}{3}}\,t$. Similarly, $E(k,t)$ would describe the structure of the small-scale eddies in a frame of reference moving with the large-scale motions, and would take the form

$$E(k,t) = E(k)f(k^{\frac{2}{3}}\epsilon^{\frac{1}{3}}t).$$

The over-all effect of these modifications would be that a quantity of the order of $[kE(k)]^{\frac{1}{2}}$ would appear in place of v_0 in the expression (3.60) for $K(r)$. Kraichnan (1959, § 9.1) has shown that this substitution resolves the conflict between the Kolmogorov theory of turbulence and the theory based on the direct-interaction approximation. The quantity $[kE(k)]^{\frac{1}{2}}$ may be considered as the r.m.s. velocity associated with the motions of wave-numbers k as they are convected by the large-scale motions. Substitution of this quantity for v_0 in (3.60) gives

$$K(r) \propto \epsilon^{\frac{1}{2}(n-1)}\, v_0^{\frac{1}{2}(5-3n)}\, r^{\frac{1}{2}(n+1)}, \tag{3.71}$$

which, taking $n = \frac{5}{3}$, leads to (3.70). With these changes in the interpretation of $\tilde{G}(k,t)$ and $E(k,t)$, the sensitivity of our results to the form of the inertial range spectrum remains.

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